

# Math 1B Discussion Section SOLUTIONS

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems and not just the answers.

1. Find the radius and interval of convergence for each of the following power series:

(a)  $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

(b)  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

We'll use the root test:  $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(x-2)^n}{n^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x-2|}{n} = 0$ , which is less than 1 for all  $x$ , so  $R = \infty$  and  $I = (-\infty, \infty)$ .

(c)  $\sum_{n=1}^{\infty} x^n \ln\left(1 + \frac{1}{n}\right)$  We'll use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \ln\left(1 + \frac{1}{n+1}\right)}{x^n \ln\left(1 + \frac{1}{n}\right)} \right| \stackrel{LH}{=} \lim_{n \rightarrow \infty} \left| \frac{x \frac{1}{1+\frac{1}{n+1}} \left(-\frac{1}{(n+1)^2}\right)}{\frac{1}{1+\frac{1}{n}} \left(-\frac{1}{n^2}\right)} \right| = \lim_{n \rightarrow \infty} \left| \frac{x \frac{1}{\frac{(n+1)^2+(n+1)}{(n+1)}}}{\frac{1}{\frac{n^2+n}{n^2+n}}} \right| = \lim_{n \rightarrow \infty} \left| x \frac{n^2+n}{(n+1)^2+n+1} \right| = |x|$$

So we just want  $|x| < 1$ . That is,  $-1 < x < 1$ . Now we test the endpoints:

$(x = -1)$ :  $\sum (-1)^n \ln\left(1 + \frac{1}{n}\right)$  is an alternating series with  $b_n = \ln\left(1 + \frac{1}{n}\right)$  and thus converges since  $b_n \rightarrow 0$  and is decreasing.

$(x = 1)$ :  $\sum \ln\left(1 + \frac{1}{n}\right) = \sum \ln\left(\frac{n+1}{n}\right) = \sum \ln(n+1) - \ln(n)$ , which is a telescoping series. We write out a few terms to see that

$$s_n = \ln 2 - \ln 1 + \ln 3 - \ln 2 + \ln 4 - \ln 3 + \cdots + \ln(n+1) - \ln n = -\ln 1 + \ln(n+1) = \ln(n+1)$$

Then  $\lim s_n = \infty$  and thus the series does not converge.

Now that we've tested the endpoints, we can see that  $R = 1$  and  $I = [-1, 1)$

(d)  $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n4^n}$

As usual, we do the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(3x-1)^{n+1}}{(n+1)4^{n+1}} \frac{n4^n}{(3x-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x-1)n}{4(n+1)} \right| = \frac{|3x-1|}{4} = \frac{3|x-\frac{1}{3}|}{4}$$

So we want  $3|x-\frac{1}{3}|/4 < 1$ , which is the same as  $|x-\frac{1}{3}| < 4/3$ , so  $R = 4/3$ , and the center of our interval is  $a = 1/3$ . To find the endpoints, we re-write this inequality as  $-4/3 < x-1/3 < 4/3$ , and add  $1/3$  to everything to get  $-1 < x < 5/3$

$(x = -1)$ : The series is then  $\sum \frac{(-4)^n}{n4^n} = \sum \frac{(-1)^n}{n}$ , which converges by the alternating series test.

$(x = 5/3)$ : We have  $\sum \frac{4^n}{n4^n} = \sum \frac{1}{n}$ , which is divergent by the  $p$ -series test.

Therefore,  $I = [-1, 5/3)$  and  $R = 4/3$

2. Given the interval of convergence for  $\sum a_n x^n$ , can you determine the radius? How about the other way around?

Given an interval, the radius is just half the length of it, which is really easy to calculate (Note: this is why we call it the radius of convergence—just like the radius of a circle is half the distance across, the radius of an interval is half its length). Given just a radius, you don't know anything about the endpoints and thus can't determine the interval.

3. True/False: For those that are true, explain why. For those that are false, give a counterexample.

- (a) If  $\sum a_n(-3)^n$  converges, then  $\sum a_n 2^n$  converges.

True. Consider the power series  $\sum a_n x^n$ . Then we know there is some number  $R$  such that it converges whenever  $|x| < R$  and diverges whenever  $|x| > R$ . Note that  $\sum a_n(-3)^n$  being convergent is the same as saying that this power series converges at  $x = -3$ . Then we know that  $R$  must be at least 3. Then we must have that  $\sum a_n x^n$  converges whenever  $|x| < 3 \leq R$ , so in particular it converges at  $x = 2$ , which is just  $\sum a_n 2^n$ .

- (b) If  $\sum a_n(-3)^n$  converges, then  $\sum a_n 3^n$  converges.

False.  $\sum \frac{x^n}{n3^n}$  converges at  $x = -3$ , but diverges at  $x = 3$ .

- (c) If  $\sum a_n x^n$  has a positive radius of convergence, then  $\lim a_n = 0$

False. Consider  $\sum x^n$ , which has radius of convergence 1. But each  $a_n$  is just 1, so the limit is too.

4. For each  $n$ , let  $f_n(x) = \frac{\sin(nx)}{n^2}$ . Show that  $\sum f_n(x)$  converges for all  $x$ . Is the same true of the series of derivatives,  $\sum f'_n(x)$ ?

a)  $\frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}$ , so by the comparison test,  $\sum \frac{|\sin(nx)|}{n^2}$  converges and thus  $\sum \frac{\sin(nx)}{n^2}$  converges absolutely (and thus converges).

b)  $f'_n = \frac{n \cos nx}{n^2} = \frac{\cos(nx)}{n}$ . Note for  $x = 0$ , this is just  $\frac{1}{n}$ , so the series  $\sum f'_n$  diverges at  $x = 0$  (actually, any multiple of  $2\pi$  would have worked here).

By now, you've seen that you can differentiate power series term-by-term and the result still converges on the same interval (except maybe the endpoints). This example shows that that is a special property of power series, and is not true of a series made out of just any functions.