Instructions

- Work through the following review problems as a group
- Make sure to focus not just on getting the correct answers, but also on how you would actually write your proofs/solutions
- Feel free to skip around–there's way more problems here than the actual midterm will have, so focus on whatever your group wants practice with.
- As always with review/practice tests, the inclusion or exclusion of certain topics should not be taken as an indication of what will be on the actual midterm.
- These problems are meant to be a **supplement** to the worksheets, quizzes, and homeworks you've had so far.

Logic, Sets, Functions

1. Find a truth table for the compound proposition $(\neg p \to q) \vee r \to (\neg q \wedge r)$ and determine whether it is a tautology.
2. Find the converse and the contrapositive for the statement "If it is cloudy, then it is raining"
3. Determine whether each of the following logical statements are true. The domain for all quantifiers is $\mathbb{N}$
   (a) $\forall x \exists y (x \leq y)$
   (b) $\exists y \forall x (x \leq y)$
   (c) $\forall x \exists y (y \neq x \wedge x \equiv y \pmod 5)$
   (d) $\forall x (13 \not| x \to \exists y (xy \equiv 1 \pmod{13}))$
   (e) $\forall x (6 \not| x \to \exists y (xy \equiv 1 \pmod 6))$
4. Suppose $\sim$ is an equivalence relation on the set A which has countably many equivalence classes, each of which is countable. Show that A is countable.
5. Prove that if A is uncountable and $A \subseteq B$ then B is uncountable.
6. Prove that $\log_7 15$ is irrational.
7. Prove that $A - (A - B) = A \cap B$ for any sets A, B
8. Prove that if $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$

Number Theory

1. Find $\gcd(142, 76)$ along with integers s, t such that $142s + 76t = \gcd(142, 76)$
2. Show that there are no solutions in positive integers to $x^2 - 5y^2 = 2$
3. Show that a number is divisible by 7 iff the sum of its octal (ie, base 8) digits is also divisible by 7.
4. Find all solutions to the system of congruences $\begin{cases} x \equiv 1 \pmod 2 \\ 2x \equiv 3 \pmod 5 \\ 3x - 1 \equiv 2 \pmod 7 \end{cases}$
5. Find $6^{43} \mod 11$
6. Prove that if $ac \equiv bc \pmod m$, then $a \equiv b \pmod{m/\gcd(m, c)}$
7. Prove that the system of congruences $\begin{cases} x \equiv 2 \pmod 6 \\ x \equiv 3 \pmod 9 \end{cases}$ has no solutions. Why is this not a contradiction to the CRT?
8. Consider the relation R on $\mathbb{Z}$ given by $aRb \Leftrightarrow \gcd(a, b) = 1$. Determine whether this relation is symmetric, reflexive, transitive and/or antisymmetric.

Induction & Recursion

1. Suppose $f : \mathbb{N} \to \mathbb{N}$ has the property that $f(1) = 1$ and $f(n) = f(n-1) + (2n-1)$. Find a formula for $f(n)$ and prove your answer is correct using mathematical induction.
2. Use induction to prove that 9 divides $n^3 + (n+1)^3 + (n+2)^3$ whenever n is a nonnegative integer
3. Prove that $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$
4. The Lucas numbers are defined by $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$. Prove that $F_n + F_{n+2} = L_{n+1}$ where F_i denotes the usual fibonacci sequence
5. Show that if you draw n circles in the plane, each pair of which intersect in exactly 2 points and no three of which meet in a common point, then you must necessarily divide the plane into $n^2 - n + 2$ regions
6. Find the general solution to $a_n = -a_{n-1} + 6a_{n-2} + n$
7. Find a recurrence relation for the number of ways to tile a $1 \times n$ board using 1×1 and 1×3 rectangles, each of which is available in 3 colors.
8. Prove that if n, b are positive integers, then the base-b expansion of n is unique.

Counting

1. How many relations on a set of n elements are there that are
   (a) Reflexive?
   (b) Irreflexive?
   (c) Symmetric?
   (d) Neither reflexive nor irreflexive?
2. How many ways can you rearrange the letters in the work PEPPERCORN?
3. What is the largest possible set of integers you can select from the the set $\{1, 2, 3, \ldots, 2008\}$ such that no two elements of your set sum to 2009?
4. How many different ways can you put k distinguishable books on r bookshelves?
5. How many ways can you pick a box of 13 bagels from 20 varieties if
   (a) there are no restrictions
   (b) all bagels must be the same
   (c) there must be at least 5 poppyseed bagels
   (d) you must pick at least two different varieties
6. Find a generating function for the number of ways to make n cents using pennies, nickels, dimes, and quarters.
7. How many solutions in non-negative integers are there to $x_1 + x_2 + x_3 = 30$ with $x_1 \geq 4, x_2 \leq 10, x_3 \leq 12$
8. Find a generating function for the number of solutions in positive integers to $2x + 3y + 2z = n$. Now do the same for $2x^2 + y + 3z = n$

Probability

1. What is the probability that a randomly generated bit string of length 10 is a palindrome?
2. Suppose you have two bins filled with red and green balls the first of which has 5 red and 3 green balls and the second has 8 red and 10 green balls.
   (a) If you put on a blindfold and pick randomly from one of the two bins, what is the probability of getting a red ball?
   (b) If you get a red ball, what's the probability that you picked from the first bin?
   (c) If you combine the two bins and then pick a ball at random, what's the probability of getting a red ball?
   (d) Suppose you pick 3 balls from the second bin. What's the probability that you get only red balls if i) you don't put balls back after picking, ii) you replace each ball after picking it?

3. Suppose that A, B are events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$.

   (a) What are the biggest and smallest possible values of $P(A \cap B)$ and $P(A \cup B)$?

   (b) Give examples to show all these extremes are possible

4. Suppose you deal five cards to four players from a standard 52 card deck. What is the probability that all 4 players received an ace?

5. What is the expected number of times you need to roll two dice to get a sum of 7?

6. Let X be the number of times a bias-p coin comes up heads when you toss it n times. Find $E[X]$ and $V(X)$.

7. Find an upper bound on the probability that the number of heads deviates from the mean by more than $5\sqrt{n}$ when you flip a bias-p coin n times.

8. Suppose you roll 6 fair dice. Find an upper bound on the probability that the sum is greater than 24.

9. Find the variance in the number of fixed points for a randomly generated permutation on n elements. Hint: $(X_1 + X_2 + \cdots + X_n)^2 = \sum X_i^2 + 2 \sum i \neq j X_i X_j$

Relations

1. Determine whether each of the following relations are reflexive, symmetric, transitive, and/or antisymmetric. Justify your answers

   (a) $xRy \Leftrightarrow x + y = 0$ on $\mathbb{Z}$

   (b) $xRy \Leftrightarrow |x - y| \in \mathbb{Q}$ on $\mathbb{R}$

   (c) $xRy \Leftrightarrow x > y^2$ on $\mathbb{R}$

   (d) $xRy \Leftrightarrow x|2y$ on $\mathbb{Z}$

2. Determine which (if any) of the above relations are partial orders or equivalence relations. Find the equivalence classes for the equivalence relations.

3. Prove that $\cong$ is an equivalence relation on the set of all graphs.

4. Draw a Hasse diagram and find the maximal, minimal, greatest, and least elements for the poset given by $(\{2, 3, 4, 5, 10, 20, 25, 40, 200\}, |)$

5. Show that if $(P, \preceq)$ is a poset, then so is $(P, \trianglelefteq)$ where $a \trianglelefteq b \Leftrightarrow b \preceq a$

6. Show that the relation "is a subgraph of" is a partial order on the set of all finite graphs

7. (a) Prove that in any rooted tree, the relation $\preceq$ on the set of vertices given by $v \preceq w$ iff "v is a descendent of w" is a partial order.

   (b) What are the maximal elements?

   (c) What are the minimal elements?

   (d) What is the greatest element?

   (e) Under what conditions does a least element exist?

   (f) Show that the relation $v \preceq w$ iff "depth(v) $\leq$ depth(w)" is not necessarily a partial order

8. Prove that the transitive closure of a symmetric relation is symmetric

9. A partition of a set S is a set of pairwise disjoint sets whose union is S. P_1 is called a refinement of P_2 if every set in P_1 is entirely contained in some set in P_2. Show that if $\mathbf{R}(S)$ is the set of all partitions of S, then the relation $\preceq$ defined by $P_1 \preceq P_2$ iff P_1 is a refinement of P_2 is a partial order on $\mathbf{R}(S)$

Graphs and Trees

See recent worksheets–we've been doing a lot of this recently so hopefully it's still fresh in your mind.