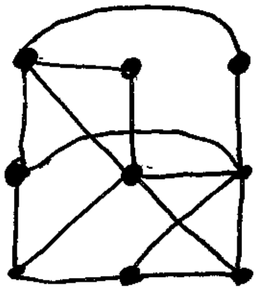
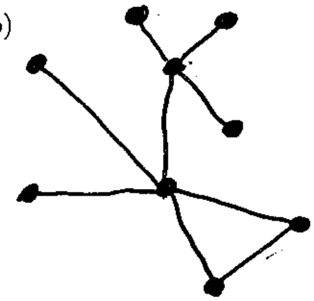
Spanning Trees

1. Find a spanning tree in each of the following graphs

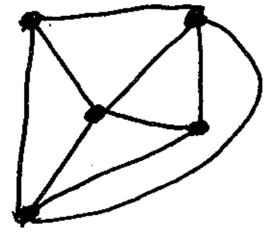
(a)



(b)



(c)



2. Find a spanning tree for

   (a) K_4 (b) Q_3 (c) C_5

3. How many nonisomorphic spanning trees do each of the following graphs have?

   (a) K_3 (b) C_5 (c) K_4

4. Prove that if e is a cut edge of G, then e must be in every spanning tree for G

5. A **forrest** is an acyclic graph (ie, multiple trees). A spanning forrest for a simple graph G is a subgraph of G that is a forrest, contains all the vertices of G, and has the same number of connected components as G.

   (a) Show that every simple graph has a spanning forrest.

   (b) How many edges are there in the spanning forrest for a graph with n vertices and c connected componenets?

6. Suppose T_1 and T_2 are two different spanning trees of the simple connected graph G. Show that if e_1 is an edge in T_1 that is not in T_2, then there is an edge $e_2 \in T_2$ such that T_1 and T_2 are both still spanning trees after you swap which T_i each e_i belongs to (ie, replace e_1 with e_2 in T_1 and vice-versa).

7. A rooted spanning tree for a directed graph is a rooted tree containing edges of the graph such that every vertex of G is the endpoint of an edge in T and all edges point "away" from the root.

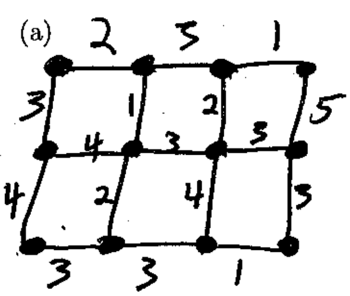
   (a) Give an example of a directed graph with no rooted spanning tree

   (b) Show that a connected digraph in which all vertices have the same in-degree as out-degree must contain a rooted spanning tree

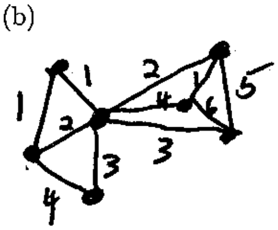
Minimum Spanning Trees

1. Find a Minimum Spanning Tree (MST) for each of the following weighted graphs

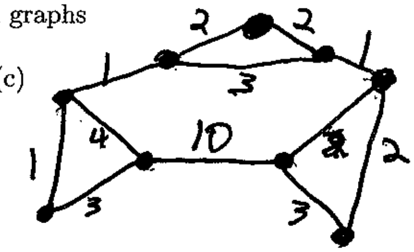
(a)



(b)



(c)



2. What does an MST represent if the nodes of your graphs represent computers, the edges represent network links, and the weights represent the cost to build those links?

3. A minimum spanning forrest for a simple weighted graph G is a spanning forrest (see above) of minimal total weight. Show that every simple weighted graph has a minimal spanning forrest.

4. Go back and do problem 6 from above if you haven't yet.

   (a) Prove that if all the weights in G are different, then there is only one MST for G

   (b) Prove that every MST for a simple graph must contain both the smallest and second-smallest weight edges.

   (c) Give an example of a graph G where the 3rd smallest edge is not a part of any MST.

   (d) Prove that if all edges incident to the vertex v have different weights, then every MST must contain the one of minimal weight.

5. The Traveling Salesman problem is a very famous problem in Computer Science and is as follows: Given a complete, weighted graph on n vertices (ie, it looks like K_n with weights), find a Hamiltonian Cycle (a cycle that visits each node exactly once) of minimum total weight. Note: this problem gets its name from the situation where you think of the graph as being a map of cities and the travel times between them.

   (a) How many Hamiltonian Cycles are there in such graph?

   (b) Show that there must be one of minimal (but not necessarily least!) weight

   (c) Explain why both Prim's and Kruskal's Algorithms are not helpful in finding it

6. Prove that every connected graph on n vertices has at least $n-1$ edges

7. So far, we've only been studying finite graphs. Let's see which results still hold in infinite graphs.

   (a) Show that if G has countably many vertices, then it also has countably many edges

   (b) Show that a connected infinite graph has a spanning tree

   (c) What falls apart in our proof that every connected weighted finite graph has a minimal spanning tree?

   (d) Show that if all edge weights are integers, then a minimal spanning tree is still guaranteed to exist.

   (e) Consider the graph G that has a vertex for each real number and an edge between the vertices representing x and y iff $|x-y| \in \mathbb{N}$. Is this graph connected? How many connected components does it have? How many vertices are in each connected component?