Partial Orders

1. Determine whether each of the following are partial orders. For those that are, decide whether or not they are linear orders.
   (a) $(\mathbb{N}, \leq)$
   (b) ("all people", "is an ancestor of") (Suppose you are considered an ancestor of yourself)
   (c) $(\{f|f : \mathbb{R} \to \mathbb{R}\}, f \prec g \Leftrightarrow \forall x\ f(x) < g(x))$
   (d) $(\mathbb{Z}, |)$
   (e) $(\mathcal{P}(\{a, b, c\}), \subseteq)$

2. Draw the Hasse diagram for the partial order given by divisibility on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. What are the maximal and minimal elements?

3. Suppose $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ are partial orders.
   (a) Show that the lexicographic order given by $(a, b) \preceq (c, d) \Leftrightarrow (a \prec_1 c \vee (a = c \wedge b \preceq_2 d))$ is a partial order on $P_1 \times P_2$
   (b) Show that the product order given by $(a, b) \prec (c, d) \Leftrightarrow a \prec_1 c \wedge b \prec_2 d$ is also a partial order

4. Consider the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |\})$
   (a) What are the maximal elements? What are the minimal elements?
   (b) Is there a least element? A greatest element?
   (c) Find a lower bound for $60, 72$.
   (d) Find all upper bounds for $2, 9$
   (e) Is this a lattice?

5. A poset is called **well-founded** if there is no infinite descending chain of elements. A poset is called **well-ordered** if every non-empty subset has a least element. Consider the poset $\mathbb{Z}, \preceq$ given by $x \prec y$ iff $|x| < |y|$
   (a) Show that this is a poset
   (b) Show that it is well-founded, but not well-ordered.

6. Show that the lexicographic order on the product of two well-founded posets is well-founded.

7. Show that the lexicographic order on the product of two well-ordered posets is a well-ordering.