Math 55 Worksheet

Closures

1. Draw the directed graph representing each of the following relations. What are the directed graphs for the symmetric closures of each of these? The transitive closures?
   (a) $\{(a,a),(a,b),(b,c),(c,b),(c,d),(d,a),(d,b)\}$
   (b) $\{(a,c),(a,b),(b,c),(c,b)\}$
2. Find a simple description of the symmetric closure of each of the following relations:
   (a) $<$ on $\mathbb{N}$
   (b) Divisibility relation on $\mathbb{Z}$
   (c) $x = |y|$ on $\mathbb{R}$
3. Find the transitive closure of each of the following relations. Your description should be reasonably simple:
   (a) "is a parent of" on the set of all people
   (b) $|x-y| = 1$ on $\mathbb{R}$
   (c) "there is a flight from x to y" on the set of all cities with airports
4. Does it make sense to talk about the antisymmetric closure of a relation? What about the irreflexive closure?
5. True/False. For those that are true, prove it. Throughout, $t(R)$ will denote the transitive closure of R.
   (a) If R is reflexive, then $t(R)$ is reflexive too.
   (b) If R is irreflexive, then $t(R)$ is irreflexive.
   (c) If R is symmetric, then $t(R)$ is symmetric too.

Equivalence Relations

1. Which of the following are equivalence relations? For those that are, determine what the equivalence classes represent.
   (a) "Divides" on $\mathbb{Z}$
   (b) $a \equiv b \pmod{n}$ on $\mathbb{Z}$
   (c) "is related to" on the set of all people
   (d) "knows" on the set of all people
   (e) "lives with" on the set of all people
   (f) $(a,b)R(c,d) \Leftrightarrow ad = cb$ on the set $\{(a,b) \in \mathbb{Z} \times \mathbb{Z} | b \neq 0\}$
2. Suppose we want to make a bracelet with three beads, each of which can be Red, White, or Blue. Let S be the set of all such bracelets.
   (a) Show that the relation R defined by aRb iff bracelet a can be obtained from b by a rotation is an equivalence relation
   (b) Find a representative of each equivalence class of R
3. Consider the relation R on $\mathbb{Z}$ defined by $aRb \Leftrightarrow a \equiv b \pmod{3}$. Suppose $[a]_R = [b]_R$ and that $[c]_R = [d]_R$. Show that $[a+c]_R = [b+d]_R$
4. How many equivalence relations are there on a set with n elements? (Hint: think back to balls in bins)
5. True/False. For those that are true, prove it. For those that are false, provide a counterexample.
   (a) The union of two equivalence relations is an equivalence relation
   (b) The intersection of two equivalence relations is an equivalence relation