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Math 55 Worksheet

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Relations

1. Determine whether each of the following relations are symmetric, reflexive, transitive, and/or antisymmetric.
   (a) $<$ on $\mathbb{N}$
   (b) $\leq$ on $\mathbb{N}$
   (c) The relation on $\mathbb{R}$ defined by $a \sim b$ iff $a - b \in \mathbb{Z}$
   (d) The relation on people given by "is a child of"
   (e) $x = y^2$ on $\mathbb{R}$
   (f) $\neq$

2. What's the probability that a randomly chosen relation on a set of size n is
   (a) Symmetric? (b) Reflexive? (c) Both?

3. A relation is called irreflexive if $(a, a) \notin R$ for any a
   (a) Give a natural example of an irreflexive relation
   (b) Give a (semi-)natural example of a relation that is neither reflexive nor irreflexive

4. Let R_1 be the relation "congruent mod 3" and R_2 be the relation "congruent mod 4," both on $\mathbb{Z}$. What are the relations
   (a) $R_1 \cup R_2$
   (b) $R_1 \cap R_2$
   (c) $R_2 - R_1$

5. True/False. For those that are true, prove it. For those that are false, provide a counterexample.
   (a) The intersection of two symmetric relations is symmetric
   (b) The union of two antisymmetric relations is antisymmetric
   (c) The intersection of two transitive relations is transitive
   (d) The union of two transitive relations is transitive

6. (a) What's wrong with the following "proof" that transitive + symmetric $\rightarrow$ reflexive.
   "Let $\sim$ be a transitive symmetric relation. Then we know $a \sim b \Rightarrow b \sim a$ so by transitivity, $a \sim a$ and thus $\sim$ must be reflexive"
   (b) Give an example of a relation that is transitive and symmetric but not reflexive, thus showing that the "result" from part (a) in fact false.

Representing Relations

1. What relation is represented by each of the following matrices? Assume the set is $\{1, 2, 3\}$ and the columns and rows are listed in increasing order
   (a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

2. Determine whether each of the relations from problem 1 are transitive, symmetric, reflexive, irreflexive and/or antisymmetric.

3. If R is a relation, then the relation R^{-1} is defined by $aR^{-1}b \Leftrightarrow bRa$. How does the matrix for R relate to that of R^{-1}? What about for $\overline{R}$, the complement of R?