Relations

1. Determine whether each of the following relations are symmetric, reflexive, transitive, and/or antisymmetric.

   (a) $<$ on $\mathbb{N}$

   $\boxed{\text{T, A}}$

   (b) $\leq$ on $\mathbb{N}$

   $\boxed{\text{R, T, A}}$

   (c) The relation on $\mathbb{R}$ defined by $a \sim b$ iff $a - b \in \mathbb{Z}$

   $\boxed{\text{S, R, T}}$

   (d) The relation on people given by "is a child of"

   $\boxed{\text{A}}$

   (e) $x = y^2$ on $\mathbb{R}$

   $\boxed{\text{A}}$

   (f) $\neq$

   $\boxed{\text{S}}$

2. What's the probability that a randomly chosen relation on a set of size n is

   (a) Symmetric?

   A relation is entirely determined by which pairs $(a, b) \in R$. For our relation to be symmetric, we need to decide whether each element of the form $(a, a) \in R$. There are n such elements, so we have 2^n ways decide whether the diagonal elements are in or not. Since our relation is symmetric, once we decide whether $(a, b) \in R (a \neq b)$, then we have also decided if $(b, a) \in R$. There are $\binom{n}{2}$ ways to pick 2 unordered elements, so we have $2^{\binom{n}{2}}$ ways to decide for each of these.

   Thus, the total probability is

   $$\boxed{\frac{2^{\binom{n}{2}+n}}{2^{n^2}}}$$

   (b) Reflexive?

   Of the n^2 pairs, n of them must be in a reflexive relation, thus we only have $n^2 - n$ pairs about which we get to choose its membership in R. The probability is thus

   $$\boxed{\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}}$$

   (c) Both?

   This is similar to (a), but we don't get to pick anything for the diagonal since it must be in R. Thus, the probability is

   $$\boxed{\frac{2^{\binom{n}{2}}}{2^{n^2}}}$$

3. A relation is called irreflexive if $(a, a) \notin R$ for any a

   (a) Give a natural example of an irreflexive relation

   $\boxed{\neq \text{ is an easy example}}$

   (b) Give a (semi-)natural example of a relation that is neither reflexive nor irreflexive

   $\boxed{x = y^2 \text{ on } \mathbb{R} \text{ would be one such example}}$

4. Let R_1 be the relation "congruent mod 3" and R_2 be the relation "congruent mod 4," both on $\mathbb{Z}$. What are the relations

   (a) $R_1 \cup R_2$

   $\boxed{aRb \Leftrightarrow (a \equiv b \pmod 3 \text{ or } a \equiv b \pmod 4)}$

(b) $R_1 \cap R_2$

$\boxed{aRb \Leftrightarrow a \equiv b \text{ (mod 12)}}$

(c) $R_2 - R_1$

$\boxed{aRb \Leftrightarrow a \equiv b \text{ (mod 4) and } a \not\equiv b\ (mod 3)}$

5. True/False. For those that are true, prove it. For those that are false, provide a counterexample.

   (a) The intersection of two symmetric relations is symmetric

   True. Let $R = R_1 \cap R_2$. Then if aRb, we know aR_1b and aR_2b so by symmetry of these, bR_ia for both i so bRa

   (b) The union of two antisymmetric relations is antisymmetric

   False. Let $R_1 =<$, $R_2 =\geq$. Then both are antisymmetric, but in $R_1 \cup R_2$ everything is related and thus is not antisymmetric

   (c) The intersection of two transitive relations is transitive

   True. Same idea as in (a)

   (d) The union of two transitive relations is transitive

   False. Let $R_1 = \{(1, 2)\}, R_2 = \{2, 3\}$. Then both are trivially transitive, but their union is not b/c $(1, 3) \notin R_1 \cup R_2$

6. (a) What's wrong with the following "proof" that transitive + symmetric $\rightarrow$ reflexive.

   "Let $\sim$ be a transitive symmetric relation. Then we know $a \sim b \Rightarrow b \sim a$ so by transitivity, $a \sim a$ and thus $\sim$ must be reflexive"

   There's no reason to think that for all a we can find a b such that $a \sim b$.

   (b) Give an example of a relation that is transitive and symmetric but not reflexive, thus showing that the "result" from part (a) in fact false.

   The empty relation (ie, nothing is related to anything else) is a good example.