Independence of R.V.'s, Chebyshev's Inequality

1. Determine whether each of the following pairs of random variables are independent or not:
   (a) X="number of heads," Y="number of tails," when three fair coins are flipped
   (b) X="sum of dice," Y="value of first die" when two fair dice are tossed
   (c) X ="sum of dice mod 2", Y="value of first die" when two fair dice are tossed

2. Suppose two professors are co-teaching a course and trying to write a midterm. If they have a bank of k^2 problems to choose from and each (independently) chooses k problems they want to put on the exam, what is the expected number of problems chosen by both professors? Hint: make a r.v.'s X_i, Y_i for whether each professor chose problem i or not.

3. Consider the experiment in which you flip a fair coin 3 times. Determine $V(X)$ and $\sigma(X)$ for each of the following:
   (a) $X =$"number of heads"
   (b) $X =$"longest consecutive run of the same result"

4. Suppose you flip a bias-p coin n times. Find an upper bound on the probability that you get more than $\frac{n}{2} + \sqrt{n}$ or less than $\frac{n}{2} - \sqrt{n}$ heads.

5. Find an upper bound on the probability that the sum of two fair dice is less than 3 or greater than 4.

6. Consider the probability space with 5 possible outcomes, a, b, c, each of which occurs with probability $\frac{1}{3}$. Suppose further that X, Y are random variables such that $X(a) = -1, X(b) = 0, X(c) = 1$ and $Y(a) = 0, Y(b) = 1, Y(c) = 0$.
   (a) Show that X, Y are not independent
   (b) Show that $E[XY] = E[X]E[Y]$. Note this shows that our theorem is not an if and only if.

7. Suppose we generate a permutation of n elements uniformly at random.
   (a) What is the expected number of fixed points?
   (b) What is the variance of the number of fixed points?
   Hint: if X_i, X_j are r.v.'s indicating the i, j are fixed by a certain permutation, what is $E[X_iX_j]$?
   (c) Find an upper bound on the probability that there are at least 3 fixed points.

More Probability

1. Suppose someone offers to pay you 2^n dollars if you can flip a fair coin n times and have your first Heads be on the nth flip. How much money should you be willing to wager on this game?

2. Suppose you repeatedly roll two dice until the sum is 6. What is the expected number of rolls?

3. Suppose Alice has n pieces of chocolate that she randomly distributes between two bins. Bob then takes the bin with more chocolate in it. What is the expected amount of chocolate that Bob takes?

4. Suppose you have an infinite supply of balls which you toss randomly into k bins. What is the expected number of tosses before each bin has at least one ball?
   Hint: let X_i be a r.v. to denote the number of tosses after $i - 1$ bins have balls until an ith bin gets a ball and find $E(\sum X_i)$