Bayes Rule

1. Suppose you have 3 fair dice and 1 loaded die that comes up 6 with probability $\frac{1}{3}$ and each other value with probability $\frac{2}{15}$. If you pick a die from this set at random, roll it twice, and get a 6 twice, what's the probability that you picked the loaded die?

2. A test for a certain disease is 99% specific (that is, 99% of the time the test will be negative for someone without the disease) and 98% sensitive (ie, 98% of the time the test will be positive for someone who has the disease).

   (a) If you test positve, what's the probability that you actually have the disease?

   (b) If you test positive twice, how likely is it that you have the disease? Assume the test results are independently distributed

3. Prove the following generalization of Bayes' Rule:

   If $F_1, F_2, \ldots, F_n$ are disjoint events of positive probability such that $F_1 \cup F_2 \cup \cdots \cup F_n = S$ and E is some event with positive probability, then

$$p(F_1|E) = \frac{p(E|F_1)p(F_1)}{p(E|F_1)p(F_1) + p(E|F_2)p(F_2) + \cdots + p(E|F_n)p(F_n)}$$

Expected Value

1. Suppose a certain lottery game requires players to correctly pick 6 numbers from a pool of 50. If order doesn't matter and numbers are not replaced, what are your expected winnings if you get $1,000,000 for correctly picking all 6 and $50,000 for correctly picking 5 of the six?

2. If you roll 3 fair dice, what is the expected value of the sum of the dice?

3. If you randomly toss n balls into k bins, what is the expected number of balls in each bin?

4. Suppose two people are flipping a coin five times to determine who will win a pile of 100 gold coins. If it comes up heads at least 3 times, player 1 wins; if it comes up tails at least three times, player 2 wins. Suppose further that after 3 flips, they loose the coin down a drain. If it had previously come up heads twice and tails once, what is the fairest way to divide the gold?[1]

5. In a randomly generated permutation, what is the expected number of fixed points?

6. Determine whether each of the following pairs of random variables are independent or not:

   (a) X="number of heads," Y="number of tails," when three fair coins are flipped

   (b) X="sum of dice," Y="value of first die" when two fair dice are tossed

   (c) X ="sum of dice mod 2", Y="value of first die" when two fair dice are tossed

7. Suppose two professors are co-teaching a course and trying to write a midterm. If they have a bank of k^2 problems to choose from and each (independently) chooses k problems they want to put on the exam, what is the expected number of problems chosen by both professors?

8. A fair coin is tossed until 3 heads or 3 tails have occured. What is the expected number of tosses?

9. Suppose we have a randomly generated handshaking party with n people (ie, there are n people and for any pair of them, the probability that they shook hands is $\frac{1}{2}$). What is the expected number of handshakes?

10. Suppose someone offers to pay you 2^n dollars if you can flip a fair coin n times and have your first Heads be on the nth flip. How much money should you be willing to wager on this game?

More probability questions

1. Suppose you have a bias-p coin. Describe a procedure you could use to simulate a fair coin.

[1] A problem similar to this is often thought of as the birth of modern probability theory. Type "pascal fermat probability" into Google if you want some more history

2. Suppose Alice has n pieces of chocolate that she randomly distributes between two bins. Bob then takes the bin with more chocolate in it. What is the expected amount of chocolate that Bob takes?

3. Suppose you have an infinite supply of balls which you toss randomly into k bins. What is the expected number of tosses before each bin has at least one ball?
   Hint: let X_i be a r.v. to denote the number of tosses after $i-1$ bins have balls until an ith bin gets a ball and find $E(\sum X_i)$