Instructions

- Work through the following review problems as a group
- Make sure to focus not just on getting the correct answers, but also on how you would actually write your proofs/solutions
- Feel free to skip around–there's way more problems here than the actual midterm will have, so focus on whatever your group wants practice with.
- As always with review/practice tests, the inclusion or exclusion of certain topics should not be taken as an indication of what will be on the actual midterm.

Induction and Recursion

1. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all positive n.
2. Consider the set of bit strings defined as followed:
   - $1 \in S$
   - If $w, v \in S$, then $w0 \in S$, and $w1v \in S$.
3. Use induction to prove that $n^2 + 3n$ is always even. (Can you think of a **much** easier proof?)
   Show that every string in S contains an odd number of $1's$
4. Find all solutions to the recurrence relation $a_n = -2a_{n-1} + 15a_{n-2} + 4n$
5. Find a recurrence relation for the number of ways to tile a $1 \times n$ board using $1 \times 1, 1 \times 2$, and 1×3 tiles.
6. Prove that every amount of postage ≥ 5 cents can be made using only 2 and 5 cent stamps.
7. Suppose you draw n lines in the plane, no two of which are parallel and no 3 of which meet in a common point.
   (a) Find a recurrence relation for R_n, the number of regions the plane is divided into. Hint: what happens when the new line crosses one of the previous ones?
   (b) Find an explicit formula for R_n
8. Show that if you draw n non-concentric circles in the plane, then you can color the resulting regions with just 2 colors so that two adjacent regions never receive the same color.
9.

Counting

1. How many strings of 12 decimal digits are there
   (a) with no restrictions?
   (b) that don't contain the same number twice in a row?
   (c) that have "12345" at the start or at the end?
   (d) that contain at least two 3's?
2. How many ways can you rearrange the letters in the word ABRACADABRA? (Note: there are 5 A's, 2 B's, 2 R's, 1 C, 1 D for a total of 11 letters)
3. How many ways are there to distribute n balls into k boxes if
   (a) Both the balls and boxes are distinguishable
   (b) The balls are identical, but the boxes are distinguishable
   (c) The balls are numbered, but the boxes are indistinguishable
   (d) Both the balls and boxes are indistinguishable
4. What is the coefficient of x^{55} in $(2 - 3x)^{74}$? In $\frac{1}{(1-x)^4}$?

5. Find the number of solutions in non-negative integers to

$$x_1 + x_2 + x_3 = 15;\ x_1 \leq 8,\ 5 \leq x_2 \leq 12,\ x_3 \neq 15$$

6. In the game of Yahtzee, players role 5 identical 6-sided dice and get points based on how many of them match or are in a row, etc. How many ways are there to get
   (a) a pair? (a pair is 2 of one number, and none of the other 3 matching)
   (b) a 3-of-kind?
   (c) a full house? (a 3-of-a-kind plus a 2-of-a-kind)
   (d) a long straight? (5 in a row)
   (e) a short straight? (4 in a row, but not 5 in a row)
   (f) a Yahtzee? (5-of-a-kind)
7. Repeat the above, but for Poker instead of Yahtzee.
8. Prove that from any set of 100 integers, you can always find 15 whose pairwise differences are all divisible by 7.
9. Prove that if you select any 51 of the first 100 positive integers, you must select a pair whose sum is 101.
10. Find a generating function for the number of ways to make $\$n$ using \$1, \$5, \$10, and \$20 bills if you must use at least one \$5 bill and no more than four \$20 bills. Write your answer in a closed form.
11. Recall that the Fibonacci numbers are defined by $F_n = F_{n-1} + F_{n-2};\ F_0 = 0, F_1 = 1$. Prove that $F_{2n} = F_n F_n + F_{n-1} F_{n+1}$
    (a) by using induction
    (b) by using a combinatorial proof based on the fact that F_n counts the number of ways to climb an n stairs taking either 1 or 2 steps at a time.