Generating Functions

1. What sequence is represented by each of the following generating functions?

   (a) $(x^2+1)^3$ (b) $\frac{1}{(1-2x^2)}$ (c) $\frac{x^9-1}{x-1}$

2. Find a generating function for each of the following sequences:

   (a) $a_n = 2$ (b) $a_n = 2^n$ (c) $a_n = n-1$

3. Find the coefficient of x^{12} in $\frac{1}{(1+x)^8}$

4. Find a generating function for the number of ways to make n cents using pennies, nickels, dimes, and quarters, where the order of the coins doesn't matter.

5. Show that $\binom{-\frac{1}{2}}{n} = \frac{\binom{2n}{n}}{(-4)^n}$

6. Find a generating function for the number of solutions in non-negative integers to $x_1 + x_2 + x_3 = k$ where $3 \leq x_1, 2 \leq x_2 \leq 10, 5 \leq x_3$ and x_3 is even.

7. Find a generating function for the number of solutions in non-negative integers to $x_1 + 2x_2 + 3x_3 = k$ (Hint: consider new variables $y_2 = 2x_2, y_3 = 3x_3$)

8. (With calculus)

   (a) Explain why the coefficient of x^n in the power series for $G(x)$ is $\frac{G^{(n)}(0)}{n!}$ where $G^{(n)}(x)$ is the *nth* derivative of G

   (b) Use (a) to find the coefficients of x and of x^2 in the power series for $\frac{1}{x-1}\frac{1}{(2x+1)^3}$

   (c) Find a formula in terms of derivatives for the number of solutions to $x_1 + x_2 + x_3 + x_4 = 20$ with x_2 even and x_3 divisible by 5

9. The generating functions we have been studying are called ordinary generating functions. Another class is called Exponential Generating Functions and are defined as follows: The Exponenetial Generating Function for the sequence $\{a_n\}$ is

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

   Note the division by $n!$. Thus, the exponenetial generating function for $1, 1, \ldots$ is $\sum \frac{x^n}{n!} = e^x$. Use this fact to find a closed form for the exponenetial generating function for each of the following sequences:

   (a) $a_n = 3^n$

   (b) $a_n = (-1)^n$

   (c) $a_n = n+1$

   (d) $a_n = \frac{1}{n+1}$ (Hint: re-index)