Structural Induction

1. What's wrong with the following "proof" that every Fibonacci number is even?

   $F_0 = 0$ is even. If F_n and F_{n-1} are even, then $F_{n+1} = F_n + F_{n-1}$ is the sum of two even numbers and thus is even. Thus, by induction, all fibonacci numbers are even.

2. Consider the set S defined recursively as follows: $0 \in S, 1 \in S, \lambda \in S$ and whenever $w \in S$, both $1w1$ and $0w0 \in S$. What is simple definition for this set?

3. Prove that in any bit string, 01 occurs at most one more time that 10.

4. Consider the set S defined by: $3 \in S$ and whenever $x, y \in S, x + y \in S$ and $x - y \in S$. Show that every element of S is divisible by 3.

5. Prove that in any well-formed formula, the parentheses are "balanced" in the following sense: If we start from 0 and scan the formula from left to right, adding 1 every time we see a "(" and subtracting 1 everytime we see a ")," then our running total will always be non-negative at every step.

Recurrence Relations

1. By "unwinding" each of the following recurrence relations, find a closed form expression for a_n

   (a) $a_n = a_{n-1} + n;\ a_0 = 0$

   (b) $a_n = (n+1)a_{n-1};\ a_0 = 1$

   (c) $a_n = -2a_{n-1};\ a_0 = 1$

   (d) $a_n = a_{n-1} + 2n + 1;\ a_0 = 1$

2. Find a recurrence relation for the number of ways to tile a $1 \times n$ board using 1×1 and 1×2 pieces. What sequence is this?

3. Suppose you have only dimes and quarters and want to put n cents into a vending machine. Find a recurrence relation for the number of different orderings of dimes and quarters that will add up to n.

4. (a) Prove that for any constants c_1, c_2, the sequence $a_n = c_1 2^n + c_2 3^n$ satisfies the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$

   (b) What are c_1, c_2 if you know $a_0 = 3, a_1 = 8$?

5. Suppose you draw n lines in the plane, no two of which are parallel and no 3 of which meet in a common point.

   (a) Find a recurrence relation for R_n, the number of regions the plane is divided into. Hint: what happens when the new line crosses one of the previous ones?

   (b) "Unwind" your relation to find an explicit formula for R_n

6. Prove that if r is a root of the polynomial $x^2 + bx + c$, then r^n is a solution to $a_n = -ba_{n-1} - ca_{n-2}$

Induction-based Puzzles

1. The 55 students of math 55 have found a large bag of gold containing 555 gold pieces. In order to divide it up, they decide to use the following system: at the first stage, the fiercest student proposes a distribution of the gold pieces and all students (including the fiercest) vote on it. If at least half (that is, 55/2) vote in favor, they use that distribution. Otherwise, the student is permanently banned from Evans Hall and the next fiercest student proposes a new distribution which now requires at least half of the remaining students' (ie, 54/2) votes to pass, etc, etc. You may assume students all prefer more gold to less, prefer not to be banished from Evans, and all else being equal like to kick other people out. How much gold does the fiercest student get? Hint: It's a lot.

2. Suppose you host a Ping-Pong tournament with n people in which everyone plays everyone else (that is, each person plays $n - 1$ games). Show that no matter what the outcomes of the matches, you can put all the participants in a line such that for all i the ith person in line beat the $i + 1$st person in line.