Math 55 Worksheet

Induction

1. Prove that $1 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

2. Find the flaw in each of the following "proofs."

   (a) **Claim:** In any group of n people, all n have the same birthday.

   Proof: We'll go by induction on n. If $n = 1$, then clearly that person has the same birthday as themself. Suppose the claim holds for groups of k people. We'll show it holds also for groups of $k + 1$. Suppose we have some group of $k + 1$ people, numbered 1 through $k + 1$. By the IH, 1 through k all have the same birthday and 2 through $k + 1$ all do too. Since there is someone shared between these two groups, all $k + 1$ must share the same birthday. Thus, by induction, the result holds for all n

   (b) **Claim:** All integers are perfect squares.

   Proof: Clearly 1 is a perfect square. Suppose the claim works for integers up to and including k. Then if we write $k + 1 = ab$, the IH tells us that $a = m^2$ and $b = l^2$ for some integers m, l. Thus, $k + 1 = m^2 l^2 = (ml)^2$ and $k + 1$ is a perfect squaqre. Thus, by induction we conclude that all integers are perfect squares.

3. Prove that the sum of the first n odd numbers is n^2

4. Using the product rule ((fg)' = f'g + fg') and the fact that $x' = 1$, use induction to prove that $(x^n)' = nx^{n-1}$

5. Let a_n be the sequence defined by $a_1 = \sqrt{2}$, $a_n = \sqrt{2a_{n-1}}$

   (a) Show that $1 < a_n < 2$ for all n

   (b) Show that $a_{n+1} > a_n$ for all n

   Note: a sequence like this is called bounded and monotone and is guaranteed to converge. We'll talk more about recursively defined sequences later.

6. The harmonic numbers are defined by $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$.

   (a) Show that $H_{2^n} \geq 1 + \frac{n}{2}$ for all n

   (b) Use your answer to (a) to show that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

Induction-based Puzzles

1. The 55 students of math 55 have found a large bag of gold containing 555 gold pieces. In order to divide it up, they decide to use the following system: at the first stage, the fiercest student proposes a distribution of the gold pieces and all students (including the fiercest) vote on it. If at least half (that is, 55/2) vote in favor, they use that distribution. Otherwise, the student is permanently banned from Evans Hall and the next fiercest student proposes a new distribution which now requires at least half of the remaining students' (ie, 54/2) votes to pass, etc, etc. You may assume students all prefer more gold to less, prefer not to be banished from Evans, and all else being equal like to kick other people out. How much gold does the fiercest student get? Hint: It's a lot.

2. Suppose you host a Ping-Pong tournament with n people in which everyone plays everyone else (that is, each person plays $n - 1$ games). Show that no matter what the outcomes of the matches, you can put all the participants in a line such that for all i the ith person in line beat the $i + 1$st person in line.