Rob Bayer

Math 55 MT1 Review

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Instructions

- Work through the following review problems as a group
- Make sure to focus not just on getting the correct answers, but also on how you would actually write your proofs/solutions
- Feel free to skip around–there's way more problems here than the actual midterm will have, so focus on whatever your group wants practice with.
- As always with review/practice tests, the inclusion or exclusion of certain topics should not be taken as an indication of what will be on the actual midterm.

Logic and Proof Techniques

1. Write a truth table for the compount proposition $(p \to q) \vee (q \wedge r) \to r$
2. Determine whether each of the following are true or false. The domain for all variables is the set of nonnegative integers. Justify your answers.
   (a) $\forall x \exists y (y > x)$
   (b) $\forall x \forall y (x^2 + y^2 \geq 2xy)$
   (c) $\exists x \forall y ((\exists z \;\; x = yz) \to (z = 1 \vee y = 1))$
   (d) $\exists x \forall y (x \neq y)$
3. Without using a truth table, show that $(p \to r) \wedge (q \vee p) \to (q \vee r)$ is a tautology
4. Prove the statement $\forall n \exists! m (m^2 \leq n < (m+1)^2)$ where the domain for all quantifiers is the set of non-negative integers
5. Show that if x^2 is irrational, then x is irrational.

Sets, Functions, Cardinality

1. Prove that $A \cap B = A$ if and only if $A \subseteq B$
2. Determine whether each of the following are injective, surjective, bijective, or nothing. Prove your answer
   (a) $f : \mathbb{N} \to \mathbb{N} \;\; f(n) = n^2$
   (b) $f : \mathbb{R} \to \mathbb{R} \;\; f(x) = 3x + 4$
   (c) $f : (0,1) \to \mathbb{R} \;\; f(x) = \frac{1}{x}$
3. Prove that if f is one-to-one, then $f(A \cap B) = f(A) \cap f(B)$ for any $A, B \subseteq dom(f)$.
   Note: this is **not** true without the assumption that f is injective.
4. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
5. Prove that if $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$
6. Show that the set of functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable.

Number Theory

1. Find each of the following. (Hint: (b) and (c) should be really quick)

   (a) $5^{23} \mod 18$

   (b) $14^{43} \mod 15$

   (c) $19^{120} \mod 31$ Hint: $120 = 30 \cdot 4$

2. Find all solutions to the system of congruences $\begin{cases} 2x \equiv 1 \pmod 3 \\ 3x \equiv 2 \pmod 5 \\ 4x \equiv 3 \pmod 7 \end{cases}$

3. Determine whether each of the following congruences have solutions:

   (a) $2x \equiv 3 \pmod 4$

   (b) $4x \equiv 1 \pmod 5$

   (c) $12x \equiv 18 \pmod{30}$

   (d) $12x \equiv 17 \pmod{30}$

4. Find $\gcd(123, 276)$ and write it as $s \cdot 123 + t \cdot 277$

5. Prove that if p is prime, then $\sqrt{p}$ is irrational.
   (Note: in fact, one can show that $\sqrt{n}$ is irrational whenever n isn't a perfect square, but it's a touch messier)

6. Prove that there are no solutions in integers to $x^3 + y^3 = 7xy + 4$.
   Hint: look mod 7 and show that any cube must be 0, 1, or 6.

7. Prove that $\log_2 3$ is irrational.

8. Prove that if $ac \equiv bc \pmod{mc}$, then $a \equiv b \pmod m$