Instructions

- Introduce yourselves! Despite popular belief, math is in fact a team sport!
- Find some blackboard space, a piece of chalk, and decide who will be your first scribe.
- Do the problems below, having a different person be the scribe for each one.
- Try to work out the problems as a group, but feel free to flag me down if you run into a wall.

Predicates and Quantifiers

1. If $P(x)$ is the statement "x > 0" and $G(x, y)$ is the statement $x^2 \geq y$, determine the truth value of each of the following

   (a) $P(1)$ (b) $G(2, -3)$ (c) $P(-1) \rightarrow G(-2, 1)$

2. Using the same predicates as above, determine the truth values of each of the following statements if the domain is the set of all real numbers.

   (a) $\forall x P(x)$ (b) $\exists x P(x) \wedge \forall x G(x, 0)$ (c) $\exists y G(2, y)$

3. Suppose the domain of $P(x)$ consists of the integers 0,1,2,3. Re-write each of the following statements without using quantifiers:

   (a) $\forall x P(x)$ (b) $\exists x\,(x \neq 1 \wedge P(x))$ (c) $\forall x\,(x \neq 1 \rightarrow \neg P(x))$

4. Let $S(x)$ be the statement "x is a student," $L(x)$ be "x lives in Germany" $G(x)$ be "x speaks German." Translate each of the following into English or into logic symbols as appropriate. The domain is the set of all people.

   (a) $\exists x\,(L(x) \wedge S(x))$
   (b) $\forall x(L(x) \wedge \neg S(x) \rightarrow G(x))$
   (c) $\exists x(S(x) \wedge L(x) \wedge \neg G(x))$
   (d) There is a German speaking student
   (e) Not all speakers of German live in Germany
   (f) The only German residents who don't speak German are students
   (g) Some students live in Germany, but some don't.

5. Determine whether each of the following pairs of sentences are equivalent. If so, explain why. If not, give an example of predicates and domains where they differ.

   (a) $\exists x(P(x) \wedge Q(x))$; $\exists x P(x) \wedge \exists x Q(x)$
   (b) $\forall x(P(x) \wedge Q(x))$; $\forall x P(x) \wedge \forall x Q(x)$
   (c) $\exists x(P(x) \rightarrow Q(x))$; $\exists x P(x) \rightarrow \exists x Q(x)$

Nested Quantifiers and Restricted Domains

1. Let $T(x, y)$ be "x is taking y", $L(x, y)$ be "x likes y", $R(x, y)$ be "x is required to take y." If the domain for x is all students and they domain for y is all classes, translate each of the following between English and Logic:

   (a) $\forall x \exists y L(x, y)$
   (b) $\exists y \forall x \neg L(x, y)$
   (c) $\forall y \exists x R(x, y)$
   (d) $\exists x \exists y(T(x, y) \wedge \neg L(x, y))$s
   (e) Every student is required to take at least one class
   (f) Some student likes all of their current classes
   (g) Some student only likes courses they aren't required to take

2. Determne the truth value of each of the following statements. The domain is the set of all real numbers

   (a) $\forall x \exists y (x > y)$

   (b) $\exists x \exists y (x \geq y \wedge y \geq x)$

   (c) $\forall x \exists y (x = y^2)$

   (d) $\forall x \forall y \exists z (x > y \rightarrow x > z > y)$

   (e) $\exists x \exists y (x + y = 1 \wedge x - y = 3)$

   (f) $\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (|x - 3| < \delta \rightarrow |x^2 - 9| < \epsilon)$

3. Re-write each of the following so that negation symbols appear only immediately in front of predicates.

   (a) $\neg \exists x \forall y P(x, y)$

   (b) $\neg \exists x \neg P(x)$

   (c) $\neg \forall x \forall y (P(x, y) \rightarrow \neg Q(x, y))$

   (d) $\neg (\forall x (\neg P(x) \wedge Q(x)) \vee \exists y \forall z (R(y, z) \wedge S(y, z)))$

4. The symbol $\exists! x P(x)$ stands for "there exists one and only one x such that P(x) is true" and is often pronounced "there is a unique x..." Show that $\exists! x P(x)$ can be rewritten using just regular quantifiers.

Logic Puzzles

1. While walking through the woods on the Island of Knights and Knaves you come to a fork in the trail. One path leads to a deep dark pit of doom and the other leads to a boat that can take you home. There are no signs indicating which direction to take, but there is one inhabitant of the island there waiting for you. Given that you don't know if he's a Knight or a Knave, what **one** question would you ask him to determine the correct path?

2. For their final exam, an evil math 55 professor has put a small mark on the forehead of 55 of his 555 students and will fail the entire class unless they can determine which of them has the mark. The students are not allowed to talk or communicate with each other in any way nor do they know how many of them were marked, but they can see each others foreheads. Once per minute, the Professor asks for all the marked students to come to the front of the room. If any unmarked students come, or if more than 1 hour passes, he will fail the entire class. At what minute do the 55 marked students turn themselves in?

3. Consider the following set of four statements:

   (a) One of these statements is false

   (b) Two of these are false

   (c) Three of these are false

   (d) Four of these are false

   Which of the above, if any, are true?

4. (Not really logic per-se, but kinda cool) Two bicyclists enter opposite ends of a 100-foot long tunnel that is only wide enough for one bike. One is travelling 10 ft/s and the other is travelling 5 ft/s in the opposite direction. A bird flying 20 ft/s enters the tunnel just in front of the 10 ft/s cyclist. When the bird gets to the other cyclist, it immediately turns around and flies back towards the 10 ft/s cyclist. When the bird gets back to him, it turns around again, etc, etc. How many feet does the bird fly before the bicyclists collide?

5. Four people travelling at night come to a small footbridge and need to cross to the other side. Unfortunately, they only have 1 flashlight and the bridge can only hold the weight of two people at once. One person takes 10 minutes to cross, another 5, another 2, and the last takes 1 minute. Anyone crossing must have the flashlight and when travelling together, they must go the pace of the slower person. Can they all get safely accross in 17 minutes?