Name: ______________________________

Math 55 Quiz 7 SOLUTIONS
July 22, 2009
GSI: Rob Bayer

You have until 4:00 to complete this quiz. You must show your work.

1. (3 pts) In poker, a "straight" consists of five cards with consecutive ranks and of any suits ($8\clubsuit, 9\diamondsuit, 10\spadesuit, J\clubsuit, Q\diamondsuit$ for example). Assume Aces are considered higher than Kings and straights may not "wrap around" from Aces back to 2's.

   How many ways are there to make a straight from a standard 52-card deck? Be sure to at least briefly justify your answer.

$$\underbrace{9}_{\text{starting rank}} \cdot \underbrace{4^5}_{\text{suits}}$$

2. (3 pts) Find the number of solutions in non-negative integers to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 55 \quad \text{with } x_2 \geq 5$$

   This is the same as the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 50$, which we know is

$$\binom{50+5-1}{5-1} = \binom{54}{4}$$

3. (4 pts) Prove that

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$

   Note that this is a special case of Vandermonde's Identity, and you may only use that theorem if you prove it here.

   Consider two sets, S and T that are disjoint and each have n elements in them. Then the LHS counts the number of ways to choose n elements from $S \cup T$.

   We can also count this same quantity as follows: For each k between 0 and n, there are $\binom{n}{k}\binom{n}{n-k}$ ways to choose k elements from S and the remaining $n-k$ elements from T. Thus, the total number of ways is also $\sum \binom{n}{k}\binom{n}{n-k}$ which is exactly the RHS.