Name: ______________________________

Math 55 Quiz 5 SOLUTIONS

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You have until 4:00 to complete this quiz. You must show your work.

1. (5 pts) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

   We'll go by induction on n:

   - BC: If $n = 1$, then the LHS is $1^2 = 1$ and the RHS is $\frac{1(1+1)(2+1)}{6} = 1$
   - IH: Suppose $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$
   - IS: We want to show this works for $k+1$ as well:

   $$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &\overset{IH}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

   as was to be shown.

2. (5 pts) Consider the set S defined recursively as follows:

   - $(1,1) \in S$
   - If $(a,b) \in S$, then $(2a, 3b) \in S$

   Prove that if $(a,b) \in S$, then $(a,b) = (2^n, 3^n)$ for some nonnegative integer n.

   We'll go by structural induction:

   - BC: $(1,1) = (2^0, 3^0)$
   - IH: Suppose $(a,b) \in S$ and $(a,b) = (2^k, 3^k)$.
   - IS: Then $(2a, 3b) = (2 \cdot 2^k, 3 \cdot 3^k) = (2^{k+1}, 3^{k+1})$

   Thus, by structural induction it works for all $(a,b) \in S$