Math 55-2 Midterm 2 SOLUTIONS

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1. (10 pts) Short answer. You need not show any work for this section
   (a) How many ways are there to put n balls in k bins if:
      i. The balls and the bins are both numbered
      $\boxed{k^n}$
      ii. The balls are indistinguishable and the bins are numbered
      $\boxed{\binom{n+k-1}{k-1}}$
      iii. The balls are numbered and the bins are indistinguishable
      $\boxed{\sum_{i=1}^{k} S(n,i)}$
      iv. Both the balls and the bins are indistinguishable
      $\boxed{p_k(n)}$
   (b) Find the coefficient of x^8 in $(5x-7)^{21}$
   $\boxed{\binom{21}{8}5^8(-7)^{13}}$
   (c) State the Generalized Pigeonhole Principle:
   $\boxed{\text{If you put } n \text{ pigeons in } k \text{ boxes, some box contains at least } \lceil \frac{n}{k} \rceil \text{ pigeons}}$
   (d) Define what it means for a permutation to be a derangement:
   $\boxed{\text{The permutation has no fixed points. (ie, every element gets moved)}}$
   (e) Find the flaw in the following "proof" that if $a \neq 0$, then $a^n = 1$ for all non-negative integers n:
   - **BC (n=0)**: $a^0 = 1$ since $a \neq 0$
   - **IH**: Suppose the claim is true for all exponents $\leq k$. (Formally: $a^i = 1$ for all $i \leq k$)
   - **IS**: $a^{k+1} = a^k \cdot \frac{a^k}{a^{k-1}} \overset{IH}{=} 1 \cdot \frac{1}{1} = 1$

   There aren't enough base cases–when trying to prove it for a_1 (ie, $k = 0$) we need an a_{-1} which we don't have a base case for. (In general, since we go down 2 steps in our IS, we need 2 base cases)

2. (15 pts) Prove that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ for every $n \geq 1$

   We'll go by induction on n:
   - **BC (n=1)** The LHS is just $\frac{1}{3}$ and the RHS is $\frac{1}{2+1} = \frac{1}{3}$
   - **IH** Suppose that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$
   - **IS**

$$
\begin{aligned}
\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2k+1)(2k+3)} &\overset{IH}{=} \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)}\\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)}\\
&= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}\\
&= \frac{k+1}{2k+3}
\end{aligned}
$$

Thus, by induction, the result holds for all $n \geq 1$

3. (15 pts) Consider the set S defined as follows:

   - $001 \in S$
   - Whenever $w, v \in S$, $100w \in S, 00w1 \in S$ and $wv \in S$

   Prove that every string in S ends with a 1 and contains twice as many 0's as 1's.

   We'll go by structural induction:

   - **BC** "001" has both the desired properties.
   - **IH** Suppose $w \in S$ and $v \in S$ both have the desired properties. In particular, suppose both end with 1's and the w has n 1's and $2n$ 0's and that v has m 1's and $2m$ 0's.
   - **IS**
     - $100w$ ends with a 1 since w does and has $n+1$ 1's and $2n+2 = 2(n+1)$ 0's.
     - $00w1$ clearly ends with a 1 and has the same number of 0's and 1's as above.
     - wv ends with a 1 since v does and has $n+m$ 1's and $2n+2m = 2(n+m)$ 0's.

   Thus, by structural induction, every element of S has the desired properties.

4. (15 pts)

   (a) (5 pts) Find a recurrence relation for the number of ways to tile a $1 \times n$ board with 1×1 and 1×2 tiles if there are three different colors of 1×1 tiles and four different colors for the 1×2's.

   Any such tiling must start with one of 3 1×1's and then be followed by a tiling of $n-1$ or start with one of 4 1×2's and then be followed by a tiling of $n-2$. Thus, our recurrence relation is:

   $\boxed{a_n = 3a_{n-1} + 4a_{n-2};\ a_0 = 1,\ a_1 = 3}$

   (b) (10 pts) Find the general solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} - 8n + 52$

   For the homogeneous part, we sovle $r^2 - 6r + 9 = 0$ and get $(r-3)^2 = 0$, so $r = 3$ is a double root. For the non-homogeneous part, we guess $p_n = An + B$:

$$
\begin{gathered}
An + B - 6(A(n-1)+B) + 9(A(n-2)+B) = -8n + 28\\
(A - 6A + 9A)n + (B + 6A + 6B - 18A + 9B) = -8n + 28\\
4An + (-12A + 4B) = -8n + 28
\end{gathered}
$$

   Equating n coefficients gives $A = -2$, which means we need $24 + 4B = 28$, so $B = 1$.

   Thus, the general solution is

   $\boxed{a_n = C_1 3^n + C_2 n 3^n - 2n + 1}$

5. (30 pts) How many ways are there to do each of the following?

   If you want any partial credit, you must show where each term in your answer comes from/what it's trying to count.

   Note that this problem continues onto the next page

(a) Rearrange the letters in the word $WINNING$?

There are 2 I's, 3 N's, a W and a G so by the same method as the examples in the book, the answer is:

$$\boxed{\frac{7!}{2!3!}}$$

(b) Get a "2 pair" in poker? A two pair is two sets of two cards with the same rank, and one card of a different rank ($3\clubsuit, 3\heartsuit, J\diamondsuit, J\clubsuit, K\spadesuit$ for example)

$$\boxed{\underbrace{\binom{13}{2}}_{\text{ranks}} \cdot \underbrace{\binom{4}{2} \cdot \binom{4}{2}}_{\text{suits}} \cdot \underbrace{44}_{\text{other card}}}$$

(c) Create a committee of r people from a pool of m men and n women if the committee must have at least one member of each gender?

$$\boxed{\underbrace{\binom{n+m}{r}}_{\text{total}} - \underbrace{\binom{m}{r}}_{\text{only men}} - \underbrace{\binom{n}{r}}_{\text{only women}}}$$

(d) Pick non-negative integers such that $x_1 + x_2 + x_3 + x_4 = 30$ with $x_1 \leq 8$, $x_2 \leq 7$, and $x_3 \geq 5$?

The $x_3 \geq 5$ can be dealt with by solving $x_1 + x_2 + x_3 + x_4 = 25$. From work done in class, we know that there are $\binom{28}{3}$ total solutions to this. Thus, we can do the following:

$$\begin{aligned} N &= \binom{28}{3} - (\#\text{ with } x_1 \geq 9 \vee x_2 \geq 8) \\ &= \binom{28}{3} - (\#\text{ with } x_1 \geq 9) - (\#\text{ with } x_2 \geq 8) + (\#\text{ with } x_1 \geq 9 \text{ and } x_2 \geq 8) \end{aligned}$$

We can deal with each of these by solving the same equation but with $= 16, = 17, = 8$, respectively. Thus we get:

$$\boxed{\binom{28}{3} - \binom{19}{3} - \binom{20}{3} + \binom{11}{3}}$$

(e) Arrange n men and n women around a circular dinner table so that men and women alternate?

There are two main methods to solve this:

- **Method 1:** Because rotating around doesn't actually give a new arrangement, all that matters is who is to the left of Alice, who is to the left of that man, then who is to that woman's left, etc. There are n choices for which man to put to the left of Alice, then $n-1$ choices of which woman to put to the left of him, $n-1$ of which man to put next to her, etc...
  Thus, the total number is $n(n-1)(n-1)(n-2)(n-2)\cdots 1 \cdot 1 = n!(n-1)!$
- **Method 2:** Pretend for the moment that we put a special mark on one of the chairs. Then this is the same as lining up these people in a row, starting at the marked chair and going counterclockwise. We can either go $MWMW$... or $WMWM$... In either case, there are $n!n!$ ways to arrange the men and the women, so thare are $2(n!)^2$ ways of doing this.
  Since we had $2n$ choices of which chair to mark, we've overcounted everything by a factor of $2n$ and need to divide by that and get $\frac{2(n!)^2}{2n} = n!(n-1)!$

In either case, the answer is:

$$\boxed{n!(n-1)!}$$

6. (15 pts) Prove that $n\binom{n-1+m}{r-1} = r\binom{n}{r} + (r-1)\binom{n}{r-1}\binom{m}{1} + (r-2)\binom{n}{r-2}\binom{m}{2} + \cdots + \binom{n}{1}\binom{m}{r-1}$
Hint: If you're stuck, you can start by thinking of forming an r-member committee with a president.

We'll do this with a combinatorial proof:

Suppose we want to make a committee of r people from a pool of n students and m administrators and we want a committee president that is a student. We claim that both sides count this same quantity.

- **LHS:** We can first pick the president, which we have n choices for. We then have $r-1$ committee spots left to fill and $n+m-1$ people left with which to fill them. Thus, there are $n\binom{n+m-1}{r-1}$ ways to make such a committee.
- **RHS:** We can also break it down into cases based on how many students and how many administrators we want on the committee. Suppose we want k students. Then we need to fill the other $r-k$ spots with administrators. After this, since there are k students, we have k ways to pick the president. Thus, the total number is also $\sum_{k=1}^{r} k\binom{n}{k}\binom{m}{r-k}$ which is exactly the RHS.

Thus, both sides count the same thing and thus must be equal.

7. (5 pts Extra Credit) How many strings of 8 decimal digits contain at most 3 distinct digits?

Again, there's two main ways to do this:

- **First Method**

  As a first try, we might think something like this: start by choosing which 3 digits you want to use. Then for each of the 8 places, you have 3 choices, so there are 3^8 ways to fill in the digits. Thus, there are $\binom{10}{3}3^8$ ways.

  However, this overcounts the strings with only 2 or only 1 digit since 11212211 will be counted multiple times since $1, 2$ is a subset of many 3-element sets. Thus, we need to subtract off our overcounting: There are $\binom{10}{2}$ ways to pick 2 digits, each of these pairs was overcounted 7 times (each is a part of 8 triples). Thus, we might think the answer is:
  $\binom{10}{3}3^8 - 7\binom{10}{2}2^8$
  However, we now have issues with the strings with just 1 digit. There are 10 such strings, and we have so far added each of them $\binom{9}{2} = 36$ and subtracted each of them $7 * 9 = 63$ times (each is a part of 9 pairs). So we need to add back an extra 28 of them to get the count right. Thus, the total is:

  $$\binom{10}{3}3^8 - 7\binom{10}{2}2^8 + 10 * 28$$

- **Second Method**

  We can think of a string with exactly k digits as an onto function from 8 to k. We know there are $k!S(8,k)$ ways to do this. Thus, there are

  $$10 + \binom{10}{2}2!S(8,2) + \binom{10}{3}3!S(8,3)$$