Math 55-2 Final Exam SOLUTIONS

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August 14, 2009

You have until 4:00pm to complete this test. No calculators, books, notes, or consultation with other members of the class are permitted. Your exam should have 4 pages.

Unsupported or improperly supported answers will receive no credit.

1. (10 pts) Short Answer. You need not show any work for this problem

   (a) A function f is injective iff $\underline{f(x) = f(y) \Rightarrow x = y}$

   (b) A relation R is an equivalence relation iff it is symmetric, transitive, reflexive

   (c) Two events are called independent iff P(E —F) = P(E)

   (d) Circle the countable sets:

   $$\mathbb{R} \qquad \underline{\mathbb{Q}} \qquad \underline{\mathbb{Z} \times \mathbb{N}} \qquad \mathcal{P}(\mathbb{N})$$

   (e) The linear congruence $ax \equiv b \pmod{n}$ has a solution iff $\underline{\gcd(a, n) | b}$

   (f) A function $f : G_1 \to G_2$ is an isomorphism of graphs iff
   f is bijective and there is an edge b/w u and v in G_1 iff there is an edge b/w $f(u)$ and $f(v)$ in G_2

   (g) State Chebyshev's Inequality

   $$P(|X - E[X]| \geq r) \leq V(X)/r^2$$

   (h) Find a Minimal Spanning Tree in the following graph. You may simply fill in the necessary edges

   graph-sol.pdf

   (i) Find a generating function for the number of solutions in non-negative integers to $2x_1 + 3x_2 + 4x_3 = n$

   $$\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}$$

2. (15 pts)

   (a) Find a truth table for the compound proposition $(p \vee q) \to (q \wedge \neg r)$

p	q	r	$p \vee q$	$q \wedge \neg r$	$(p \vee q) \to (q \wedge \neg r)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	F	F
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	F	T

(b) Prove or provide a counterexample: $A - (B - C) = (A - B) - C$ for any sets $A, B,$ and C
This is false. If $A = B = C = \{1\}$, then
$A - (B - C) = A - \emptyset = A = \{1\}$ and $(A - B) - C = \emptyset - C = \emptyset$

3. (15 pts)

(a) Find a solution to $63x \equiv 3 \pmod{132}$
We start wit hthhe extended Euclidean Algorithm:

$$\begin{aligned} 132 &= 2 \cdot 63 + 6 \\ 63 &= 10 \cdot 6 + 3 \\ 6 &= 2 \cdot 3 + 0 \end{aligned}$$

Working backwards, we get

$$\begin{aligned} 3 &= 63 - 10 \cdot 6 \\ &= 63 - 10 \cdot (132 - 2 \cdot 63) \\ &= 21 \cdot 63 - 10 \cdot 132. \end{aligned}$$

So $21 \cdot 63 - 10 \cdot 132 \equiv 3 \pmod{132}$ and thus $21 \cdot 63 \equiv 3 \pmod{132}$ and $x = 21$ is one possible solution.

(b) Find all solutions in positive integers to $x^2 = 15 + y^2$. Be sure to explain why your solution(s) are the only ones.
This is the same as wanting all solutions of $x^2 - y^2 = 15$ which factors as $(x - y)(x + y) = 15$. Since $15 = 5 \cdot 3$, and x, y are both positive integers, the fundamental theorem of arithmetic says there are only 4 possibilities:

- $\begin{cases} x - y = 15 \\ x + y = 1 \end{cases}$ Then $2x = 16$, so $x = 8$ and $y = -7$, which is not positive.
- $\begin{cases} x - y = 1 \\ x + y = 15 \end{cases}$ Then $2x = 16$ so $x = 8$ and $y = 7$
- $\begin{cases} x - y = 5 \\ x + y = 3 \end{cases}$ Then $2x = 8$ so $x = 4$ and $y = -1$ which is not positive
- $\begin{cases} x - y = 3 \\ x + y = 5 \end{cases}$ Then $x = 4$ so $y = 3$.

Thus the solutions are $(8, 7), (4, 3)$. These are the only ones since if $x, y \in \mathbb{Z}^+$ we must have $x + y \in \mathbb{Z}^+$ and $x - y \in \mathbb{Z}$ and prime factorizations are unique.

4. (15 pts)

(a) How many ternary strings (strings of 0's, 1's, 2's) of length n have at least one of each symbol?
This is the same as "total - # missing 0 or 1 or 2." By inclusion-exclusion, this is

$$\begin{aligned} N = 3^n - (&\text{\# missing 0} + \text{\# missing 1} + \text{\# missing 2} \\ &- \text{\# missing 0 \& 1} - \text{\# missing 0 \& 2} - \text{\# missing 1 \& 2} + \\ &+ \text{\# missing 0, 1, and 2}) \\ = 3^n &- 2^n - 2^n - 2^n + 1 + 1 + 1 - 0 \\ = 3^n &- 3 \cdot 2^n + 3 \end{aligned}$$

(b) How many ways can you place n books on r shelves if

i. All the books are identical
This is the same as indistinguishable balls in distinguishable boxes, so it's $\binom{n+r-1}{r-1}$

ii. The books are all different and the order in which you place them on an individual shelf matters
We can do this by first laying all the books out on the ground in some order and then putting in $r-1$ "dividers" to indicate which books go on which shelf. There are $n!$ ways to order the books, so the total is

$$n!\binom{n+r-1}{r-1} = n!\frac{(n+r-1)!}{(r-1)!n!} = \frac{(n+r-1)!}{(r-1)!}$$

5. (10 pts) Prove that in any set of 555 integers there must be a set of 55 all of whose pairwise differences are divisible by 10.

Consider the remainders of these numbers mod 10. There are 10 possible remainders, so by the GPHP, there must be a subset of $\lceil \frac{555}{10} \rceil = 56$ that have the same remainder mod 10. Then by definition these are all $\cong$ mod 10 so by an alternative definition of modular equivalence, 10 divides all their differences.

Note: in fact, we showed that there are 56 elements with the desired property, which is even stronger than what the problem asked us to prove.

6. (10 pts) Prove that every connected simple graph on n vertices must contain at least $n-1$ edges.

Let G be a connected simple graph with n vertices. Then G has a spanning tree, which must have $n-1$ edges. Since spanning trees are subgraphs of the original graph, G must have had at least $n-1$ edges to start.

7. (15 pts) Suppose you flip a bias-p coin n times and get \$2 every time it comes up heads, but have to pay \$1 every time it comes up tails.

(a) (6 pts) What are your expected earnings?

We'll do the standard trick here and let $X_i = \begin{cases} 2 & \text{if } i^{th} \text{ flip is H} \\ -1 & \text{o/w} \end{cases}$.
Then if X is a r.v. for our earnings, we see that $X = X_1 + X_2 + \cdots + X_n$
By linearity of expectations, we have:

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} E[X_i] \\ &= \sum (2 \cdot P(\text{ith flip is H}) + (-1) \cdot P(\text{ith flip is T})) \\ &= \sum (2p - (1-p)) \\ &= n(3p-1) \end{aligned}$$

(b) (6 pts) What is the variance in your earnings?
We'll use the same variables as above. Then since all the X_i's are indpendent, we can find $V(X)$ by simply adding up all the $V(X_i)'s$.

$$\begin{aligned} V(X) &= \sum_{i=1}^{n} V(X_i) \\ &= \sum (E[X_i^2] - (E[X_i])^2) \\ &= \sum (2^2 \cdot P(\text{ith flip is H}) + (-1)^2 \cdot P(\text{ith flip is T})) - (3p-1)^2 \\ &= \sum (4p + (1-p) - 9p^2 + 6p - 1) \\ &= n(9p - 9p^2) \\ &= 9n(p - p^2) \end{aligned}$$

(c) (3 pts) What is the minimum value of p for which you should be willing to play this game?
$E[X] \geq 0$ iff $p \geq \frac{1}{3}$ so $p = \frac{1}{3}$ is the minimum value for which you should be willing to play.

8. (10 pts) Let G be a directed graph. Show that the relation $\preceq$ defined by $V_1 \preceq V_2$ iff there is a path in G from some node in V_1 to some node in V_2 is a partial order on the set of strongly connected components of G. Note: for any node, the path of length 0 counts as a path from that node to itself.

   We'll show that $\preceq$ is reflexive, transitive, and anti-symmetric. (SCC = "Strongly Connected Component")

   - (Reflexive) If V is a SCC of G, then V is non-empty and thus contains at least one node. By the "Note" this has a (trivial) path to itself and thus $V \preceq V$
   - (Transitive) Suppose $V_1 \preceq V_2 \preceq V_3$. And let v_1, v_2, u_2, u_3 be such that there is a path from v_1 to v_2 and from u_2 to u_3. Then since V_2 is strongly connected, there is a path in V_2 (and thus in G) from v_2 to u_2. Amalgamating these paths gives a path $v_1 \to v_2 \to u_2 \to u_3$ which is in fact a path from v_1 to u_3. Thus $V_1 \preceq V_3$
   - (Anti-Symmetric) Suppose U, V are SCC's of G and that $U \preceq V$ and $V \preceq U$. Then there must be $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that there is a path (in G) from u_1 to v_1 and from v_2 to u_2.

     Claim: u_1 and v_1 are strongly connected.

     Proof: By above, there is a path from u_1 to v_1 so we need only find a path from v_1 to u_1. Since V is strongly connected, there is a path from v_1 to v_2. Similarly, there is a path from u_2 to u_1. Then the path $v_1 \to v_2 \to u_2 \to u_1$ is a path from v_1 to u_1 and we're done.

     Thus, $U \cap V \neq \emptyset$ and since "strongly connected" is an equivalence relation, we conlcude $U = V$