§7.4#24

(a) Find a generating function for the number of solutions to $x_1 + x_2 + x_3 + x_4 = k$ with $x_1 \geq 3, 1 \leq x_2 \leq 5, 0 \leq x_3 \leq 4, x_4 \geq 1$

This is just

$$\begin{aligned} G(x) &= (x^3 + x^4 + \cdots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots) \\ &= x^5(1 + x + x^2 + \cdots)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + \cdots) \\ &= x^5 \frac{1}{(1-x)^2} \cdot \frac{(x^5-1)^2}{(x-1)^2} \\ &= x^5 \frac{(x^5-1)^2}{(1-x)^4} \end{aligned}$$

(b) Find the coefficient of x^7:

From the extended binomial theorem,

$$\frac{1}{(1-x)^4} = \sum \binom{-4}{i}(-1)^i x^i = \sum \binom{4+i-1}{i} x^i$$

Thus, our generating function can be written

$$G(x) = x^5(1 - 2x^5 + x^10)(\binom{3}{0} + \binom{4}{1}x + \binom{5}{2}x^2 + \binom{6}{3}x^3 + \cdots)$$

Then we can only get an x^7 term by selecting the x^5, the 1, and the x^2. Thus, the coefficient is $\binom{5}{2} = 10$

§7.5#14 Find the number of permutations of the English alphabet that don't contain FISH, BIRD, RAT.
By Inclusion-Exclusion, this is:

$$\begin{aligned} Num &= 26! - ((\text{num with rat}) + (\text{num with bird}) + (\text{num with fish})) + \\ &+ ((\text{num with rat and bird}) + (\text{num with rat and fish}) + (\text{num with bird and fish})) - (\text{num with all 3}) \\ &= 26! - 24! - 23! - 23! + 21! \end{aligned}$$