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Math 55 Homework 5 SOLUTIONS

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§4.1#10

(a) Find a formula for $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$

The first few terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...$ so we guess $\frac{n}{n+1}$

(b) We'll go by induction:

- BC: for $n = 1$, both sides evaluate to $\frac{1}{2}$
- IH: Suppose $\frac{1}{1 \cdot 2} + \frac{2 \cdot 3}{+} ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$
- IS:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \stackrel{IH}{=} \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

as was to be shown.

§4.2#14 Show that if you start with n stones in a pile an incur a cost of k_1k_2 whenever you break a pile into two piles of sizes k_1, k_2, then the total cost to break the pile of n down to single stones is $\frac{n(n-1)}{2}$.
We'll go by strong induction:

- BC: 1 stone is already broken apart, so the cost is 0, which is the same as $\frac{1(1-1)}{2}$
- IH: Suppose the formula works for piles of size $\leq k$
- IS: Suppose we have a pile of $k+1$ stones. Then if we break it into piles of sizes k_1, k_2 we incur a cost of k_1k_2. By the IH, it costs $\frac{k_1(k_1-1)}{2}$ and $\frac{k_2(k_2-1)}{2}$ to fully break apart each of these. Thus, the total cost is:

$$k_1k_2 + \frac{k_1(k_1-1)}{+}\frac{k_2(k_2-1)}{2} = \frac{2k_1k_2 + k_1^2 - k_1 + k_2^2 - k_2}{2}$$
$$= \frac{k_1^2 + 2k_1k_2 + k_2^2 - (k_1+k_2)}{2}$$
$$= \frac{(k_1+k_2)^2 + (k_1+k_2)}{2}$$
$$= \frac{(k+1)^2 - (k+1)}{2}$$
$$= \frac{(k+1)(k+1-1)}{2}$$

as was to be shown.

Thus, by strong induction, the result holds for all n.