§3.5#14

(a) Show that 6 and 28 are perfect.

  (a) The divisors of 6 are 1, 2, and 3, and their sum is in fact 6.
  (b) The divisors of 28 are 1, 2, 4, 7, and 14 and their sum is in fact 28 as well.

(b) Show that $2^{p-1}(2^p - 1)$ is perfect whenever $2^p - 1$ is prime.

The divisors here are $1, 2, 2^2, 2^3, \dots, 2^{p-1}$ and $(2^p-1), 2(2^p-1), 2^2(2^p-1), \dots 2^{p-2}(2^p-1)$. Summing these gives:

$$1 + 2 + 2^2 + \cdots 2^{p-1} + (2^p - 1)(1 + 2 + 2^2 + \cdots + 2^{p-2}) = 2^p - 1 + (2^p - 1)(2^{p-1} - 1) = (2^p - 1)2^{p-1}$$

§3.6#30 Show that a positive integer is divisble by 11 iff the difference of the sum of its even-position decimal digits and the sum of its odd-position decimal digits is divisible by 11.
Let $n = d_k 10^k + d_{k-1}10^{k-1} + \cdots + d_0$ be the decimal representation of $n$. Then since $10 \equiv -1 \ (\mathbf{mod}\ 11)$, we have:

$$n \equiv d_0 + (-1)d_1 + (-1)^2 d_2 + \cdots + (-1)^k d_k \ (\mathbf{mod}\ 11)$$

Note that $(-1)^i = \begin{cases} -1 & \text{if i is odd} \\ 1 & \text{otherwise} \end{cases}$. Thus, the sum above can be re-written as

$$n \equiv d_0 - d_1 + d_2 - d_3 + \cdots \pm d_k \ (\mathbf{mod}\ 11)$$

We note the above is the same as $(d_0 + d_2 + d_4 + \cdots) - (d_1 + d_3 + \cdots)$, so $n \equiv 0 \ (\mathbf{mod}\ 11)$ (ie, $11|n$) iff $(d_0 + d_2 + d_4 + \cdots) - (d_1 + d_3 + \cdots) \equiv 0 \ (\mathbf{mod}\ 11)$ (ie, the difference of the appropriate sums is divisible by 11).