These notes are meant as a supplement to sections 5.3 and 5.4. Most of the results can be found there, but I'm not a huge fan of how some of the material is presented there.

Definition 1. *An ordering of a set is called a permutation of that set. An ordering of k elements of a set is sometimes called a k-permutation.*

Question 1. *How many permutations of "ABCDE" are there?*

Answer 1. *This is just like problems we did in 5.1: there are five letters here, so we have five choices for what to pick first. After that, we have 4 remaining choices for the 2nd letter, then three for the 3rd, 2 for 4th, and then the 5th letter will be forced to be whatever we haven't used yet. By the product rule, there are* $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ *ways to rearrange these letters.*

Question 2. *How many 3-permutations are there of "ABCDE"?*

Answer 2. *Well, this is basically the same thing as above, except we only have to make three choices–5 options for the first, 4 for the second, and 3 for the third. This means we have* $5 \cdot 4 \cdot 3 = 60$ *different ways of choosing and ordering three letters from this set.*

If you think about it for a bit, you should be able to see that in general, the number of permutations of n objects is $n!$ and the number of k-permutations is $n(n-1)\cdots(n-(k-1))$, which can be re-written as $\frac{n!}{(n-k)!}$

The above is all well-and-good, but what if order doesn't matter? The following example should be illustrative:

Question 3. *How many ways are there to form a committee of 5 people from a group of 20?*

Answer 3. *For the moment, let's pretend the order in which we pick people for this committee matters. Then from above, we know that there are* $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = \frac{20!}{15!}$ *different ways to make such a committee. However, since order doesn't actually matter, we've drastically over counted–the committeess Alice/Bob/Charlie/Dan/Edith and Bob/Alice/Edith/Charlie/Dan are the same but we've counted them twice. In general, we've counted every possible committee* $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ *times since we know that's how many ways we can rearrage 5 people. Thus, our final answer should be* $\frac{20!/15!}{5!} = \frac{20!}{15!5!}$

Definition 2. *Picking k elements from a set in which order doesn't matter is sometimes called a k-combination.*

Using the example above, there are $\frac{20!}{15!5!}$ 5-combinations of a 20 element set.

Theorem 1. *There are* $\frac{n!}{k!(n-k)!}$ *ways to pick k elements from an n element set.*

Proof. We know that if order matters, there are $\frac{n!}{(n-k)!}$ ways to pick k elements. Since order doesn't, we note that we've overcounted each set of k because we've counted each one k factorial times (the number of permutation of that set). Thus, we divide by $k!$ and get the desired result. □

The quantity $\frac{n!}{k!(n-k)!}$ (ie, the number of ways to pick k things from an n-element set) occurs so frequently, we give it its own symbol:

Definition 3. *The symbol* $\binom{n}{k}$ *(pronounced "n choose k") stands for* $\frac{n!}{k!(n-k)!}$ *and is the number of ways to pick k elements from an n element set if order doesn't matter.*

So, continuing the example above, there are $\binom{20}{5}$ ways of making a 5-memember comittee from a 20-person group.

Question 4. *How many different ways are there to deal 4 5-card poker hands from a deck of 52 cards.*

Answer 4. *There are* $\binom{52}{5}$ *hands the first person could get. After that, there are* 47 *cards left, so there are* $\binom{47}{5}$ *hands for the second,* $\binom{42}{5}$ *for the third, and finally there are* $\binom{37}{5}$ *ways to make a hand for the 4th person. By the product rule, the total number is*

$$\binom{52}{5} \cdot \binom{47}{5} \cdot \binom{42}{5} \cdot \binom{37}{5}$$

Question 5. *How many bit strings of length 14 contain exactly 3 ones?*

Answer 5. *We have* $\binom{14}{3}$ *ways to pick which positions should contain the 1's, and then we have to fill in 0s for the rest, so the total number is just* $\binom{14}{3}$

Question 6. *In the game of poker, how many ways are there to get a full house? A full house is a 2-of-a-kind and a 3-of-a-kind.*

Answer 6. *There are 13 possibilites for the rank (2, 3, K, J, etc) for the 2-of-a-kind, and there are $\binom{4}{2}$ ways to pick the suits for those. There are 12 possibilites for the rank of the 3-of-a-kind, and there are $\binom{4}{3}$ ways to pick those suits. Thus, the total number is*

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{3}$$

Question 7. *What is the coefficient of xy^2 in $(x+y)^3$?*

Answer 7. *We can think of $(x+y)^3 = (x+y)(x+y)(x+y)$. Then if we multiply this all out, we'll get $xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$. Then there are 3 of these with one x and 2 y's, so 3 must be the coefficent of xy^2.*

What if we wanted to find the coefficent of x^3y^8 in $(x+y)^{11}$. We could do something similar to the above, but we'd get $2^{11} = 2048$ terms! However, all we really care about is those terms with 3 x's and 8 y's. That is, we want all the different ways to pick 8 positions for y's out of 11 total positions. Thus, the coefficient is $\binom{11}{3}$.
This generalizes as follows:

Theorem 2. *(Binomial Theorem)*

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0y^n$$

Definition 4. *Because of the above, quantities like $\binom{n}{k}$ are called binomial coefficients.*

Claim 1.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof. (First) By the binomial theorem, we know that

$$2^n = (1+1)^n = \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

which is exactly what we're trying to prove. □

The above is quite quick and succint, but doesn't really give any insight into **why** this should be true. Let's try a different argument:

Proof. (Second) We know 2^n is the number of subsets of an n element set. Now any such subset must have $0, 1, 2, \ldots,$ or n elements in it. From earlier work, we know that there are $\binom{n}{k}$ different subsets of size k. Thus, the total number of subsets is also equal to $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$.
Therefore, since they count the same things, these quantities must be equal. □

Proofs like this are called "combinatorial proofs" because they work by showing that you're really just counting the same thing in two different ways.

Claim 2.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof. We'll give a combinatorial proof here: The left hand side counts the number of subsets of size k of a set with $n+1$ elements. Let T be such a set and let $a \in T$ be any element. Then each size-k subset of T either contains a or it doesn't. If it does, then it must also contain $k-1$ elements of the remaining n. If it doesn't, then all k of its elements must come from the remaining n. Thus, the right hand side also counts the number of size k subsets of an $n+1$ element set (it just breaks it down into cases). □

The above is the basis for Pascal's Triangle. See book for pretty pictures. (Explain in class.)

Claim 3. *(Vandermonde's Identity)*

$$\binom{m+n}{r} = \binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \cdots + \binom{m}{0}\binom{n}{r}$$

Proof. We'll give another combinatorial proof. Suppose we want to form a committee of r people from a pool of m men and n women. Then there are $\binom{m+n}{r}$ ways to do it. We can also think of this as follows: Any such committee has either: r men and 0 women, r-1 men and 1 woman, r-2 men and 2 women.... or 0 men and r women. Since there are $\binom{m}{k}\binom{n}{r-k}$ ways to pick k men and $r-k$ women, we see we get exactly the right hand side of what we're trying to prove. □