

Proof By Picture

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Today

Integer Sums

Non-Proof

Fibonacci Identities

Geometry

Partition Numbers

Triangular Numbers Revisited

Definition

The n^{th} triangular number is denoted T_n and is defined by

$$T_n = 1 + 2 + \cdots + n$$

Theorem

$$T_n = \frac{n(n+1)}{2}$$

Triangular Numbers Revisited

Traditional Proof.

Induction on n :

▶ **BC:** ($n=1$) $T_1 = 1 = \frac{1 \cdot 2}{2}$

▶ **IH:** Suppose $1 + 2 + \dots + k = \frac{k(k+1)}{2}$

▶ **IS:**

$$\begin{aligned} 1 + 2 + \dots + k + k + 1 &\stackrel{IH}{=} \frac{k(k+1)}{2} + k + 1 \\ &= \left(\frac{k}{2} + 1\right)(k+1) \\ &= \frac{(k+2)(k+1)}{2} \end{aligned}$$

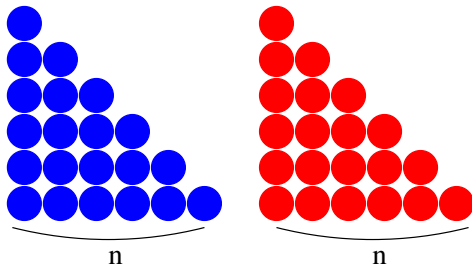


Yuk! This gives no info about **why** this should be true, and required that we knew the formula in advance.

Triangular Numbers Revisited

Representing T_n with dots

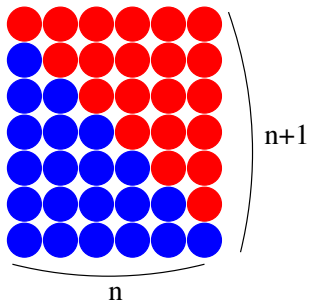
Let's try a visual representation:



T_n blue dots and T_n red dots for a grand total of $2T_n$ dots

Triangular Numbers Revisited

Proof.



$$2T_n = \text{Total Dots} = n(n+1)$$



T_{2n}

Theorem

$$T_{2n} = 3T_n + T_{n-1}$$

Traditional Proof.

$$\begin{aligned} T_{2n} &= \frac{2n(2n+1)}{2} = \frac{4n^2 + 2n}{2} \\ &= \frac{(3n^2 + 3n) + (n^2 - n)}{2} = 3\frac{n(n+1)}{2} + \frac{(n-1)n}{2} \\ &= 3T_n + T_{n-1} \end{aligned}$$



Yuk! Not only did we have to know the formula in advance, but we had to do some creative algebra in the middle.

$$T_{2n} = 3T_n + T_{n-1}$$

Induction?

Induction Proof.

▶ **BC:** $T_4 = 10 = 3 \cdot 3 + 1 = 3T_2 + T_1$

▶ **IH:** $T_{2k} = 3T_k + T_{k-1}$

▶ **IS:**

$$\begin{aligned} T_{2n} &= 1 + 2 + \dots + 2k + k + 1 + 2k + 2 \\ &\stackrel{IH}{=} 3T_k + T_{k-1} + (2k + 1) + (2k + 2) \\ &= 3T_k + T_{k-1} + (k + k + 1) + 2(k + 1) \\ &= 3(T_k + (k + 1)) + T_{k-1} + k \\ &= 3T_{k+1} + T_k \end{aligned}$$

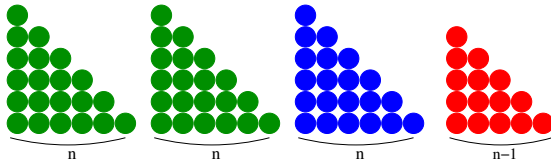


Yuk! Not only did we have to know the formula in advance, but the IS also required some creative algebra.

Plus, we still don't really know **why** this should be true.

T_{2n} Representing $3T_n + T_{n-1}$ with dots

Let's try a visual representation:



And if we cleverly rearrange these...

Plus, while doing this proof we also showed:

Corollary (of the proof)

$$T_n + T_{n-1} = n^2$$

The Main Point

Not only are visual proofs sometimes easier than traditional proofs, they can also help explain **why** a certain result is true.

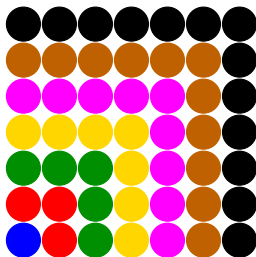
which is of course the hallmark of a good proof.

Sum of the Odd Integers

Theorem

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Proof.



Sum of Squares

Definition

Let S_n be the sum of the first n perfect squares. So

$$S_n = 1^2 + 2^2 + 3^2 + \dots + n^2$$

Theorem

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

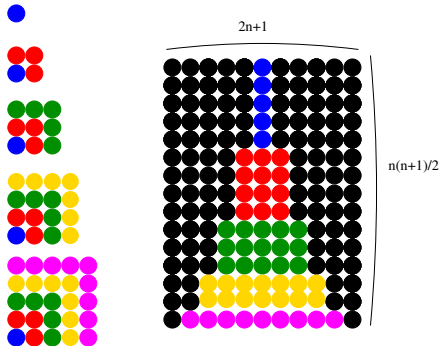
Traditional Proof.

Messy induction—see old quiz solutions.



Sum of Squares

Elegant Proof.



$$2S_n \text{ Black Dots} + S_n \text{ Color Dots} = 3S_n \text{ Total Dots} = \frac{n(n+1)(2n+1)}{2}$$

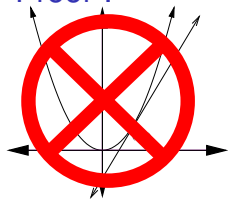


When a Picture Isn't a Proof

Claim

The curves $y = x^2$ and $y = 2x - 1$ cross exactly once.

“Proof”.

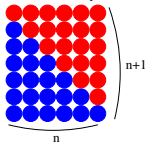


This depends on the ability to draw these graphs **perfectly!**

Guidelines

In general, a picture is a proof only if:

1. The picture represents an abstract idea



$$(1 + 2 + \dots + n) + (n + (n - 1) + \dots + 1) = \\ (n + 1) + (n + 1) + \dots + (n + 1) = n(n + 1)$$

2. The specific drawing of the picture isn't actually important
3. The picture can be “scaled up” to as big an n as necessary

Remember: it's not the picture that's the proof—it's the **idea** that the picture is representing that really counts

The Fibonacci Numbers

Definition

The Fibonacci numbers are defined by

$$F_0 = F_1 = 1; F_{n+2} = F_{n+1} + F_n$$

Sum of Squares of Fibonacci Numbers

Theorem

$$F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

Traditional Proof.

1. **BC** ($n=0$): $F_0^2 = 1 = 1 \cdot 1 = F_0 F_1$

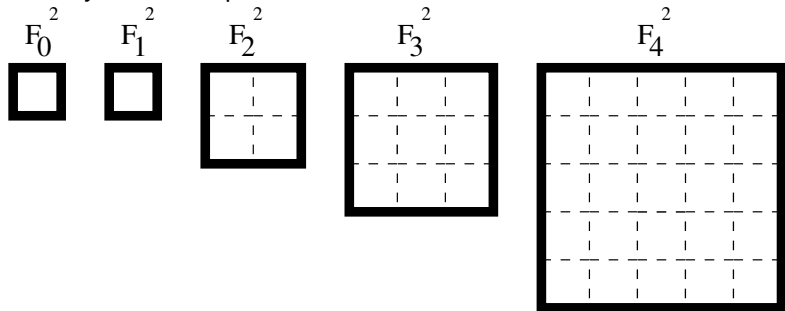
2. **IH**: $F_0 + \cdots + F_k^2 = F_k F_{k+1}$

3. **IS**:
$$\begin{aligned} F_0 + \cdots + F_k^2 + F_{k+1}^2 &\stackrel{IH}{=} F_k F_{k+1} + F_{k+1}^2 \\ &= F_{k+1} (F_k + F_{k+1}) \\ &= F_{k+1} F_{k+2} \end{aligned}$$



Not bad, but still not very enlightening. Still required that we knew the formula in advance.

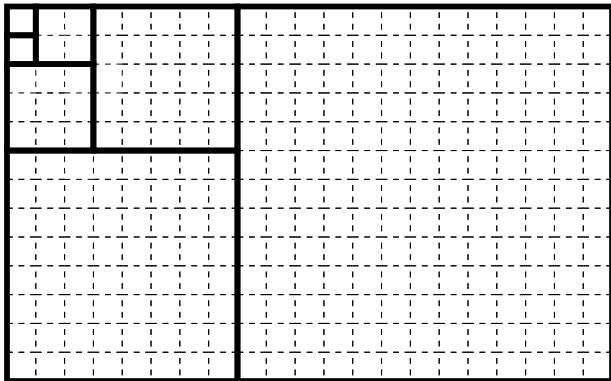
Let's try a visual representation:



Now let's see if we can arrange these cleverly

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$$

Visual Proof.



$$F_0^2 + F_1^2 + \cdots + F_n^2 = \text{Total Area} = F_n(F_n + F_{n-1}) = F_n F_{n+1}$$



Theorem

$$F_{n+m} = F_n F_m + F_{n-1} F_{m-1}$$

Induction Proof.

Induction on what? n ? m ?

Yuk!



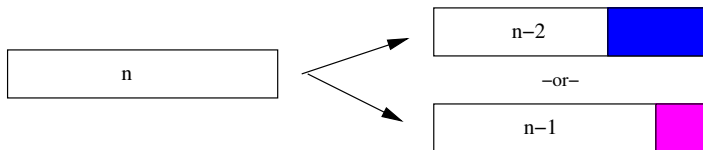
Fibonacci Numbers as Tilings

Recall:

Theorem

There are F_n ways to tile a $1 \times n$ board with 1×1 and 1×2 rectangles.

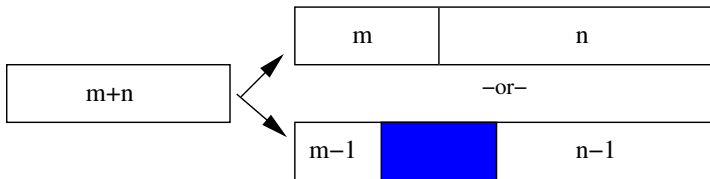
Proof.



Theorem

$$F_{n+m} = F_n F_m + F_{n-1} F_{m-1}$$

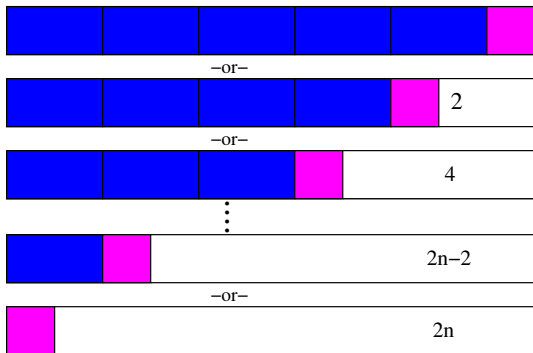
Tiling Proof.



Theorem

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}$$

Proof.



What about things that aren't integers?

Can visual proofs possibly work for theorems involving real numbers?

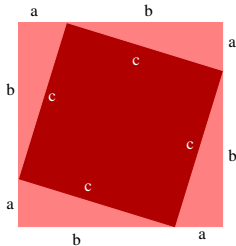
The Pythagorean Theorem

Theorem

If c is the hypotenuse of a right triangle with sides a and b , then

$$a^2 + b^2 = c^2$$

Proof.

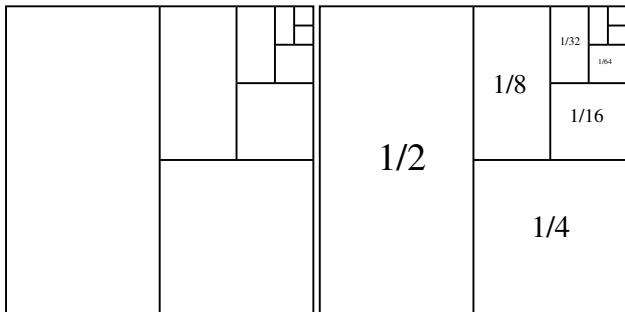


$$\begin{aligned}c^2 &= \text{Area}(\text{red square}) = (a+b)^2 - 4 \left(\frac{ab}{2}\right) \\ &= a^2 + 2ab + b^2 - 2ab \\ &= a^2 + b^2\end{aligned}$$



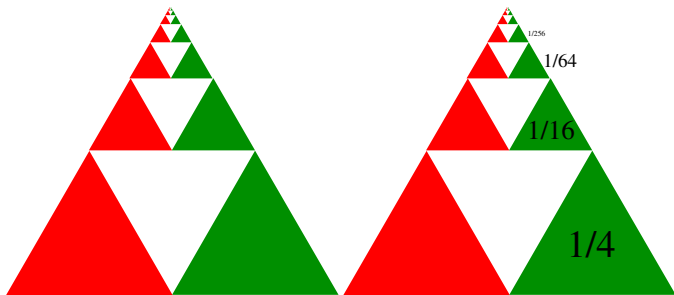
$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots = 1$$

Proof.



$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots = \frac{1}{3}$$

Proof.



$$\frac{1}{3} = \text{Green Area} = \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots$$



Partitions

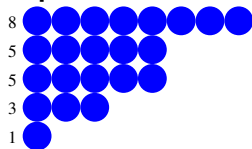
Definition

A partition of n is a non-increasing list of integers that sum to n .

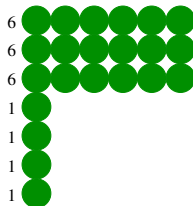
Examples

$$22 = 8 + 5 + 5 + 3 + 1$$

In pictures:



$$22 = 6 + 6 + 6 + 1 + 1 + 1 + 1$$



FYI, pictures like this are called Ferrers diagrams

Theorem

The number of partitions of n into k parts is the same as the number of partitions of n whose largest part has size k

Examples

With $n = 6$ and $k = 3$ we have:

3 parts

$$2 + 2 + 2$$

$$3 + 2 + 1$$

$$4 + 1 + 1$$

Biggest piece is 3

$$3 + 3$$

$$3 + 2 + 1$$

$$3 + 1 + 1 + 1$$

With $n = 7$, $k = 2$ we have:

2 parts

$$6 + 1$$

$$5 + 2$$

$$4 + 3$$

Biggest piece is 2

$$2 + 1 + 1 + 1 + 1 + 1$$

$$2 + 2 + 1 + 1 + 1$$

$$2 + 2 + 2 + 1$$

Theorem

The number of partitions of n into k parts is the same as the number of partitions of n whose largest part has size k

Traditional Proof.

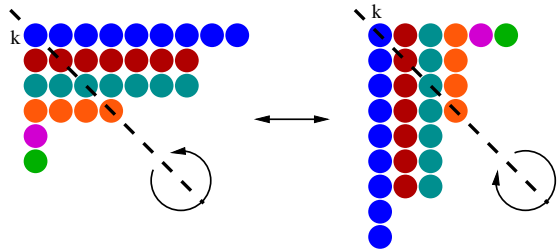
Where would you even start?



Theorem

The number of partitions of n into k parts is the same as the number of partitions of n whose largest part has size k

Proof.



So there is a bijection between these two types of partitions and thus there are the same number of each.



References



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