The Method of Undetermined Coefficients

1. For each of the following non-homogeneous equations, write the form of your guess for y_p. Don't worry about solving for the actual coefficients in this problem:

   (a) $y'' + y = x^2 e^{2x}$

   (b) $y'' + 2y' + y = x\cos 4x$

   (c) $y'' - 4y' + 13y = e^{2x}\cos 3x$

   (d) $y'' - 4y' + 13y = e^{2x}\sin 3x + xe^{2x}\cos 3x + e^x$

   (e) $y'' - 6y' + 9y = (x^2+1)e^{3x}$

   (f) $y'' - y = 3e^{-x} + \cos x + 5$

2. Find the general solution to each of the following:

   (a) $y'' + 2y' - 3y = e^{2x}$

   (b) $y'' + 2y' + y = e^{-x}\cos x$

   (c) $y'' + 2y' + y = xe^x$

   (d) $y'' - 8y' + 16y = e^{4x}$

3. Solve the initial value problem $y'' + y' - 2y = e^x;\;\; y(0) = 0, y'(0) = 1$

4. (Why we multiply by x sometimes). Consider the differential equation $y'' + 2y' - 3y = e^x$.

   (a) What should you use for y_p?

   (b) What happens if you don't realize that $r = 1$ is a root and accidentally used $y_p = Ae^x$?

5. Consider the equation $y'' - 5y' + 6y = xe^x + \cos x$. Note that you will have 4 undetermined coefficients if you try to solve this directly. In order to avoid this mess, it's sometimes easier to solve equations like this separately and then add the results. Find the general solution to this equation by first finding particular solutions to $y'' - 5y' + 6y = xe^x$ and $y'' - 5y' + 6y = \cos x$ and then adding them.

Extra Problems These Problems are a little harder but also a little more interesting than the ones above. Don't feel bad about asking for help on these as they're meant to be challenging.

1. (e is irrational) In lecture, you should have seen Taylor's Remainder Formula, which says that for any infinitely differentiable function f and any number x, $f(x) = \left(\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right) + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}$ where z is some number between x and a. Let's use this formula (which won't be on the quizzes or final) to prove that e is irrational:

   (a) By using $N = q, x = 1, a = 0$ in Taylor's Remainder Formula, show that
   $$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{e^z}{(q+1)!}$$
   where z is some number between 0 and 1.

   (b) Show that if $e = \frac{p}{q}$ for some integers p, q (ie, if e were rational) then $q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}))$ must be an integer.

   (c) Use part (a) to show that $q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!})) = \frac{e^z}{q+1}$ for some $z \in (0,1)$

   (d) Use part (c) to show that $0 < q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!})) < 1$
   Hint: You may assume $q > 1$ since if $q = 1$, then saying $e = \frac{p}{q}$ would say that e is an integer which is clearly false.

   (e) Explain why (b) and (d) show that e cannot be written as $\frac{p}{q}$ for any integrs p, q and thus must be irrational.

2. (Asymptotic Behavior)

   (a) Show that if a, b, c are all greater than 0, then all solutions to $ay'' + by' + cy = 0$ have the property that $\lim_{x\to\infty} y(x) = 0$

   (b) If $a > 0, c > 0$ but $b = 0$, show that the result from part (a) is no longer true, but that all solutions are bounded as $x \to \infty$. Bounded means that the solutions have a limited range of values.

   (c) If $a > 0, b > 0$ and $c = 0$, show that all solutions approach some constant (not necessarily 0) as $x \to \infty$. Determine this constant in terms of $y(0)$, which we'll call a, and $y'(0)$, which we'll call b