Second Order Homogeneous Differential Equations

1. Which of the following second order differential equations are linear? Homogeneous?
   (a) $e^x y'' + \cos(3x^2)y' + 3y = 0$
   (b) $y'' + 3y' + 7y = \cos x$
   (c) $y'' + 3xy' + y^2 = 0$
   (d) $\tan(y'') + \cos(x)y' = e^x$

2. Find the general solution to each of the following differential equations:
   (a) $2y'' + 3y' - 2y = 0$
   (b) $y'' + 10y' + 25y = 0$
   (c) $y'' - y' + 13y = 0$

3. Solve each of the following initial/boundary value problems, or show that no solution can exist:
   (a) $y'' + 4y' + 4y = 0;\ \ y(0) = 1, y'(0) = 3$
   (b) $y'' = y;\ \ y(0) = 0, y(2) = 2$
   (c) $y'' + 4y = 0;\ \ y(0) = 1, y(\pi) = 0$
   (d) $y'' - 6y' + 13y = 0;\ \ y(0) = 4, y'(0) = 0$

Extra Problems These Problems are a little harder but also a little more interesting than the ones above. Don't feel bad about asking for help on these as they're meant to be challenging.

1. (Using ODEs to prove trig identities)
   Consider the differential equation $y'' = -y$
   (a) Find the general solution to this equation
   (b) Show that $y = \cos(x + a)$ is a solution for any constant a.
   (c) Argue that parts (a) and (b) show that $\cos(x + a) = C_1 \cos x + C_2 \sin x$ for an appropriate choice of C_1 and C_2
   (d) Without using trig identities, find C_1 and C_2. (Hint: what should $y(0)$ and $y'(0)$ be? Remember that a is a constant, so C_1 and C_2 can refer to it.) Does this formula look familiar? (If not, try changing a to A and x to B)

2. (Euler's Formula)
   (a) By writing out the corresponding Maclaurin series, show that $e^{ix} = \cos x + i \sin x$
   (b) Using (a), show that $e^{\alpha+\beta i} = e^{\alpha}(\cos\beta + i\sin\beta)$
   (c) Show that $e^{i\pi} = -1$

3. (Why it all works)
   (a) Show that if r is a root of $ax^2 + bx + c = 0$, then e^{rx} is a solution to $ay'' + by' + cy = 0$
   (b) Show that if y_1 and y_2 are solutions to $ay'' + by' + cy = 0$, then $C_1y_1 + C_2y_2$ is also a solution
   (c) Show that if r is a double root of some polynomial $p(x)$, then r is also a root of $p'(x)$. In particular, show that if r is a double root of $ax^2 + bx + c = 0$, then r is also a root of $2ax + b = 0$.
   Hint: r being a double root means that $p(x)$ can be factored as $(x - r)^2q(x)$
   (d) Use part (c) to show that if r is a double root of $ax^2 + bx + c = 0$, then xe^{rx} is a solution to $ay'' + by' + c = 0$

4. (e is irrational) In lecture, you should have seen Taylor's Remainder Formula, which states that the error in using the degree-n Taylor Polynomial (ie, the Taylor Series, but cut off at the $(x - a)^n$ term) to approximate f at some point x is given by $\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ where z is some number between x and a. Let's use this formula (which won't be on the quizzes or final) to prove that e is irrational:

(a) By using $n = q, x = 1$ in Taylor's Remainder Formula, show that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{e^z}{(q+1)!}$$

where z is some number between 0 and 1.

(b) Show that if $e = \frac{p}{q}$ for some integers p, q (ie, if e were rational) then $q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}))$ must be an integer.

(c) Use part (a) to show that $q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!})) = \frac{e^z}{q+1}$ for some $z \in (0, 1)$

(d) Use part (c) to show that $0 < q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!})) < 1$

Hint: You may assume $q > 1$ since if $q = 1$, then saying $e = \frac{p}{q}$ would say that e is an integer which is clearly false.

(e) Explain why (b) and (d) show that e cannot be written at $\frac{p}{q}$ for any integrs p, q and thus must be irrational.