Math 1B PDP Worksheet

Finding Sums

1. Find a closed form (ie, no $\sum$ or "$\cdots$") for each of the following. For those that include an x, your answer should be a function. For those without an x, you answer should be a number.

   (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{3^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty}n(n+1)x^{n+2}$

   (d) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^{n+2}(n+2)(n+1)}$

   (e) $\displaystyle\sum_{n=1}^{\infty}(n+2)x^{n+1}$

Taylor Series Tricks

1. For each of the following f, n, a triples, find $f^{(n)}(a)$:

   (a) $f(x) = \arctan x;\ a = 0;\ n = 19$

   (b) $f(x) = \sin(x^2);\ a = 0;\ n = 12$

   (c) $f(x) = \dfrac{1}{3+x};\ a = -2;\ n = 8$

   (d) $f(x) = x\cos(2x);\ a = 0;\ n = 14$

   (e) $f(x) = \dfrac{x-1}{(3-x)^2};\ a = 1;\ n = 11$

2. (e is irrational) In lecture, you should have seen Taylor's Remainder Formula, which states that the error in using the degree-n Taylor Polynomial (ie, the Taylor Series, but cut off at the $(x-a)^n$ term) to approximate f at some point x is given by $\dfrac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ where z is some number between x and a. Let's use this formula (which won't be on the quizzes or final) to prove that e is irrational:

   (a) By using $n = q, x = 1$ in Taylor's Remainder Formula, show that

   $$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{e^z}{(q+1)!}$$

   where z is some number between 0 and 1.

   (b) Show that if $e = \frac{p}{q}$ for some integers p, q (ie, if e were rational) then $q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}))$ must be an integer.

   (c) Use part (a) to show that $q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!})) = \frac{e^z}{q+1}$ for some $z \in (0,1)$

   (d) Use part (c) to show that $0 < q!(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!})) < 1$
   Hint: You may assume $q > 1$ since if $q = 1$, then saying $e = \frac{p}{q}$ would say that e is an integer which is clearly false.

   (e) Explain why (b) and (d) show that e cannot be written at $\frac{p}{q}$ for any integrs p, q and thus must be irrational.