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# Math 1B PDP Worksheet

April 14, 2009

**Finding Power Series**

1. Find a power series representation for $\dfrac{x^2}{(1-2x)^2}$ and find its radius and interval of convergence.

2. Find a power series representation for $\dfrac{x}{(1-x^2)^2}$ by
   (a) Recognizing this as being close to the derivative of $\frac{1}{1-x^2}$. What is the interval of convergence?
   (b) Recognizing this as a substitution and a multiplication by $x$ of $\frac{1}{(1-x)^2}$ which you found in part (e) above. What is the interval of convergence?
   (c) Are your answers the same?

3. What function is represented by each of the following power series?
   (a) $\displaystyle\sum_{n=0}^{\infty} x^n$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$
   (c) $\displaystyle\sum_{n=1}^{\infty} n x^{2n-1}$ [Hint: what if it were $2nx^{2n-1}$ inside the sum?]

4. (a) What is $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$?
   (b) Use a power series for $\dfrac{x}{(1-x)^2}$ to find $\displaystyle\sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n$
   (c) Use a power series for $\frac{x+x^2}{(1-x)^3}$ to find $\displaystyle\sum_{n=0}^{\infty} n^2 \left(\frac{1}{2}\right)^n$

5. Find the sum of each of the following series:
   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n n!}$
   (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!4^n}$
   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(4n+2)x^{4n+1}}{(2n+1)!}$
   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(\ln 2)^{2n}}{n!}$

6. You saw in class that differentiating a power series term-by-term doesn't change the radius or interval of convergence, except possibly at the endpoints. Let's show that this is a property unique to power series:
   (a) Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is absolutely convergent for all $x$.
   (b) Show that the series obtained by differntiating term-by-term diverges whenever $x = 2\pi, 4\pi, 6\pi, \ldots$
   (c) Why does this not contradict the theorem about differentiating power series term-by-term?

7. When we say $f(x) = \sum_{n=0}^{\infty} a_n x^n$, what we really mean is that these two things give the same value no matter what you plug in for $x$ as long as $x$ is in the interval of convergence.
   (a) In particular, when we plug in $x = 0$, both sides should give the same result. Based on this, what must $a_0$ be?
   (b) If all the values of the power series and the function are the same, then all derivatives should be the same too (to see this, think about the formal definition of derivative). Using this, how can you determine what $a_1$ must be? $a_2$?
   (c) Find a general formula for $a_n$. If you've taken Calculus BC before, does this look familiar?