Finding Power Series

1. Starting from the geometric series, find a power series representation for each of the following functions, and determine the radius of convergence. DO NOT do a Taylor expansion.

   (a) $\dfrac{1}{1+x^3}$

   (b) $\dfrac{1}{4+x^2}$

   (c) $x\tan^{-1}(2x)$

   (d) $\dfrac{\ln(1+x^3)}{x}$

   (e) $\dfrac{1}{(1-x)^2}$

2. Find a power series representation for $\dfrac{x}{(1-x^2)^2}$ by

   (a) Recognizing this as being close to the derivative of $\frac{1}{1-x^2}$

   (b) Recognizing this as a substitution and a multiplication by x of $\frac{1}{(1-x)^2}$ which you found in part (e) above.

   (c) Are your answers the same?

3. What function is represented by each of the following power series?

   (a) $\displaystyle\sum_{n=0}^{\infty} x^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}x^n$

   (c) $\displaystyle\sum_{n=1}^{\infty} nx^{2n-1}$ [Hint: what if it were $2nx^{2n-1}$ inside the sum?]

4. (More proof that infinity is very, very strange)

   (a) Find a power series for $\dfrac{x}{(1-x)^2}$

   (b) Use your answer to (a) to find $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n$?

   (c) What is $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$? Does your answer to (b) seem correct based on this?

5. You saw in class that differentiating a power series term-by-term doesn't change the radius or interval of convergence, except possibly at the endpoints. Let's show that this is a property unique to power series:

   (a) Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is absolutely convergent for all x.

   (b) Show that the series obtained by differntiating term-by-term diverges whenever $x = 2\pi, 4\pi, 6\pi, \ldots$

   (c) Why does this not contradict the theorem about differentiating power series term-by-term?

6. When we say $f(x) = \sum_{n=0}^{\infty} a_n x^n$, what we really mean is that these two things give the same value no matter what you plug in for x (as long as it's in the interval of convergence).

   (a) In particular, when we plug in $x = 0$, both sides should give the same result. Based on this, what must a_0 be?

   (b) If all the values of the power series and the function are the same, then all derivatives should be the same too (to see this, think about the formal definition of derivative). Using this, how can you determine what a_1 must be? a_2?

   (c) Find a general formula for a_n. If you've taken Calculus BC before, does this look familiar?

7. Show that if $\sum a_n x^n$ has infinitely many coefficients that are non-zero integers, then the radius of convergence is at most 1. Note: this is a hard problem to prove formally–if you're getting stuck with the symbols, try to figure out intuitively why this must be true.

   Hint 1: power series are absolutely convergent on any closed interval contained in their interval of convergence.

   Hint 2: It may be helpful to prove that any sub-series of an absolutely convergent series also converges.