

Alternating Series

1. For each of the following, determine (a) whether it is an alternating series and (b) if it converges.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$

(d) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos(\pi n)}{\sqrt{n}}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n^3 + 3}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$

(f) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n} + 7}{n + 2}$

2. We've now seen how to deal with series with only positive terms, and series with alternating sign terms. How would you approach testing a series with only negative terms?

3. (a) Show that even though $\lim b_n = 0$, the series $\sum_{n=2}^{\infty} (-1)^n b_n$, where $b_n = \begin{cases} \frac{2}{n-1} & \text{if } n \text{ is odd} \\ \frac{1}{(n/2)^2} & \text{if } n \text{ is even} \end{cases}$, diverges.

(b) Show that even though b_n is decreasing, the series $\sum (-1)^n b_n$, where $b_n = \frac{n+1}{n}$, diverges

(c) Show that even though b_n is decreasing and $\lim b_n = 0$, the series $\sum \frac{1}{n}$ diverges.

4. Give an example of each of the following situations:

(a) $\sum a_n$ converges, but $\sum a_n^2$ diverges

(b) $\sum a_n$ diverges, but $\sum a_n^2$ converges

Absolute Convergence/More series problems

1. Determine whether each of the following are absolutely convergent, conditionally convergent, or divergent:

(a) Problems (a), (c), (e), (f) from problem 1 in the previous section

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(d) $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$

(c) $\sum_{n=4}^{\infty} \frac{\sin n + \cos(n^2)}{n^2 + \ln n}$

(e) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n \ln n \ln \ln n}$

2. For what values of p is $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ (a) absolutely convergent, (b) conditionally convergent, (c) divergent. Be sure to state which tests you are using to come to each of your conclusions.

3. True/False. For those that are true, prove it. For those that are false, give a counterexample:

(a) If $\sum a_n$ is absolutely convergent, then $\sum a_n^2$ converges.

(b) If $\sum a_n$ and $\sum b_n$ are conditionally convergent, then $\sum a_n b_n$ is too.

(c) If $\sum |a_n|$ diverges, then $\sum a_n$ diverges

(d) If $\sum |a_n|$ converges, then $\sum a_n$ converges

More Series Practice

1. Determine whether each of the following converges or diverges. For those that converge, **find the value of the series**

(a) $\sum_{n=1}^{\infty} \tan^{-1}(n+1) - \tan^{-1}(n)$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^3 - 4n + 5}$

(b) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{2n-1}}$

(d) $\sum_{n=1}^{\infty} \sqrt[n]{5}$

$$(e) \sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-3}}$$

$$(f) \sum_{n=1}^{\infty} \frac{e^{\sqrt{n}}}{n^3}$$

More Fun Series/Sequence Facts

Only try these problems once you've finished everything above. I'll probably keep putting these same problems on the worksheets until a lot of people have done them, so no need to rush through them today.

1. (Zeno's Paradox) Suppose you are 1 meter away from a wall and want to walk up and touch it. Then you must first go half the distance to the wall, which takes some positive amount of time, then half of the remaining distance, then half of what still remains, etc, etc. Show that despite this, it only takes a finite amount of time to walk across the room. Assume you can move at 1m/s.
2. Prove that $\overline{.9} = 1$
3. (A closed form for the Fibonacci sequence) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$
 - (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
Hint: $x^{n+1} = xx^n$
 - (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n .
 - (c) Is it even obvious that your closed form evaluates to an integer?