Math 1B PDP Worksheet

Limit Comparison Test

1. Determine whether each of the following series converge or diverge.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{3n^3+4n}{2^n(n^3+6n^2)}$

   (b) $\displaystyle\sum_{n=10}^{\infty} \frac{2n+3}{\sqrt[3]{n^5+3n^4-6n^2+1}}$

   (c) $\displaystyle\sum_{n=2}^{\infty} 1-\cos^2\left(\frac{1}{\sqrt{n}}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} \ln(\sqrt{n}+1)-\frac{1}{2}\ln(n)$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^{2n}(2^{1/n}-1)}$

   (f) $\displaystyle\sum_{n=1}^{\infty} 1-\cos\left(\frac{1}{n}\right)$

   (g) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[2^n]{2}-1\right)$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln\left(\frac{n+1}{n}\right)\sin\left(\frac{1}{\sqrt{n}}\right)n^n}{(n+1)^n}$

2. Explain why the limit comparison test doesn't work for $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ or for $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. How would you solve these problems?

3. Let p_n denote the nth prime number. For example, $p_1=2, p_2=3, p_3=5, p_4=7, p_5=11$, etc. A very important (and difficult!) theorem in number theory states that $\displaystyle\lim_{n\to\infty} \frac{p_n}{n\ln n}=1$.

   (a) Use this to show that $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots$ diverges to ∞.

   (b) Explain why part (a) shows that there must be infinitely many prime numbers

More Series Practice

1. Determine whether each of the following converges or diverges. For those that converge, **find the value of the series**

   (a) $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}(n+1)-\tan^{-1}(n)$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{2n-1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^3-4n+5}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{5}$

   (e) $\displaystyle\sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-3}}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\sqrt{n}}}{n^3}$

More Fun Series/Sequence Facts

Only try these problems once you've finished everything above. I'll probably keep putting these same problems on the worksheets until a lot of people have done them, so no need to rush through them today.

1. (Zeno's Paradox) Suppose you are 1 meter away from a wall and want to walk up and touch it. Then you must first go half the distance to the wall, which takes some positive amount of time, then half of the remaining distance, then half of what still remains, etc, etc. Show that despite this, it only takes a finite amount of time to walk across the room. Assume you can move at 1m/s.

2. Prove that $.\overline{9}=1$

3. (Infinitely many prime numbers, part 2)

   (a) Consider the series you would get by multiplying out $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)$. In terms of their prime factorization, what numbers would appear as denominators?

   (b) Do the same for $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)(1+\frac{1}{5}+\frac{1}{25}+\cdots)$

   (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)(1+\frac{1}{5}+\frac{1}{25}+\cdots)(1+\frac{1}{7}+\frac{1}{49}+\cdots)(1+\frac{1}{11}+\frac{1}{121}+\cdots)\cdots$ Does this converge or diverge?

(d) Use the fact that each multiplicand has finite value (what is it?) to show that there must be infinitely many prime numbers.

4. (A closed form for the Fibonacci sequence) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$. Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

   (c) Is it even obvious that your closed form evaluates to an integer?