Rob Bayer **Math 1B PDP Worksheet** March 17, 2009

Integral Test

1. Determine whether each of the following series converge or diverge. DO NOT use the series comparison test.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n}$
   (b) $1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{9+x^6}$
   (d) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)\sqrt{\ln\ln n}}$
2. For which values of x does $\displaystyle\sum_{n=1}^{\infty}(\ln x)^n$ converge? And for $\displaystyle\sum_{n=1}^{\infty}(\ln n)^x$?
3. Consider the series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt[3]{n-3}}$. Re-index this series to turn it into a p-series and determine if it converges or diverges.

If you finish either section early, move on to the "crazy facts" section below.

Comparison Test

1. Determine whether each of the following series converge or diverge.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-3}{n^3+4n+2}$
   (b) $\displaystyle\sum_{n=4}^{\infty} \frac{e^n}{2^n-\ln n}$
   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin^2 n}{n^2+\ln n}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+e^n}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$
   (f) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$
   (g) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{n^2-2}{n^3+5}\right)^2$
2. (a) Show that if $\sum a_n$ is a convergent series with non-negative terms, then $\sum a_n^2$ is also convergent.
   (b) However, if $\sum a_n$ is a convergent series with non-negative terms, $\sum \sqrt{a_n}$ could either converge or diverge. Give an example for each of these possibilities.
   (c) Show that if $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms then $\sum a_n b_n$ converges too.

Crazy Facts

1. Find the flaw in the following "proof" that 0=1:

$$\begin{aligned}
0 &= 0+0+0+\cdots \\
&= (1-1)+(1-1)+(1-1)+\cdots \\
&= 1-1+1-1+1-1+\cdots \\
&= 1+(-1+1)+(-1+1)+\cdots \\
&= 1+0+0+0+\cdots \\
&= 1
\end{aligned}$$

2. The Cantor Set is a set of real numbers constructed as follows: start with the interval $[0,1]$, and remove the middle third of it. That is, remove the interval $(\frac{1}{3},\frac{2}{3})$, leaving $[0,\frac{1}{3}],[\frac{2}{3},1]$. Now remove the middle third of each of these remaining intervals, leaving $[0,\frac{1}{9}],[\frac{2}{9},\frac{1}{3}],[\frac{2}{3},\frac{7}{9}],[\frac{8}{9},1]$. After continuing this process infinitely many times, you will be left with the Cantor set.

(a) Show that the total length of all the intervals you remove is 1.

(b) Convince yourselves that despite this, the cantor set has infinitely many numbers in it. Give some examples of these numbers.

(c) (Side note: it actually turns out that the Cantor Set is uncountable, meaning there are exactly the same number of numbers in it as there were in the interval $[0, 1]$ before you started removing middle thirds. The proof of this is actually very easy, but requires some knowledge of binary and ternary decimal systems. Talk to me if you're curious.)

3. Recall that the Fibonacci sequence is defined as $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

(a) If you haven't before, write out a few terms of this sequence to get a sense of the pattern.

(b) Show that $\frac{1}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_n F_{n+1}}$

(c) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1$

(d) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = 2$