More Sequences

1. Determine whether each of the following sequences converge or diverge. For those that converge, find the limit:

   (a) $\dfrac{n!}{n^n}$

   (b) $\dfrac{\cos^2 + n}{2^n + 3^n}$

   (c) $\dfrac{1}{1+(-1)^n}$

   (d) $\sqrt{n^2+1} - \sqrt{n^2-1}$

   (e) $\dfrac{n}{n+3^{(-1)^n}}$

   (f) $\left(\cos\dfrac{1}{n}\right)^{n^2}$

   (g) $\dfrac{n^{(-1)^n}}{n+\ln n}$

   (h) $n^{\frac{\ln 2}{1+\ln n}}$

   (i) $\dfrac{1+2^n}{1+(-2)^n}$

   (j) $\left(\dfrac{n}{n+1}\right)^n$

2. Prove, using the $\epsilon - N$ definition of a limit, that if $\lim a_{2n} = \lim a_{2n+1} = L$ (ie, the even and odd subsequences converge to L), then $\lim a_n = L$

3. Consider the sequence $a_n = \frac{1}{n(n+1)}$, and define a new sequence by $s_n = a_1 + a_2 + \cdots + a_n$.

   (a) What are s_1, s_2, s_3, and s_4?

   (b) Find a simple formula for s_n. Hint: partial fractions.

   (c) Does the sequence s_n converge? If so, find its limit.

   (d) Explain why your work in (c) shows that $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{=a_1+a_2+a_3+\cdots} = 1$

   (Note: this shows that $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = 1$, despite having infinitely many positive terms)

4. In addition to defining sequences by a formula of the form $a_n = f(n)$, we can also define a sequence by giving the value of a_1 and a rule for getting the next term in the sequence from the previous one. For example, we could say $a_n = 3 + a_{n-1}; a_1 = 2$. This then defines the sequence
$a_1 = 2, a_2 = 3 + a_1 = 3 + 2 = 5, a_3 = 3 + a_2 = 8, a_4 = 3 + a_3 = 11$, etc. Such a definition is called recursive. A definition that doesn't refer to previous terms or have any "..."s in it is called a closed form. For example, a closed form for the sequence above would be $a_n = 2 + 3(n-1)$

   For each of the following recursively defined sequences, find a closed form for a_n. Use your answer to determine if each is convergent or divergent.

   (a) $a_n = na_{n-1};\ a_1 = 1$

   (b) $a_n = a_{n-1} + n;\ a_1 = 1$

   (c) $a_n = ra_{n-1};\ a_1 = a$ where a, r are contants and $|r| < 1$

   (d) $a_n = \ln n + a_{n-1};\ a_1 = 0$

5. (Do the previous problem before this one)

   (a) Explain (or prove, if you feel up to it) why $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ for any convergent sequence.

   (b) Find a recursive definition for the sequence $\{a_n\}$ given by $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$

   (c) Use (a) and (b) to find $\lim_{n\to\infty} a_n$, assuming this limit exists[1].

6. (hard) Prove, using the $\epsilon - N$ definition of a limit, that if $\lim a_{2n} \neq \lim a_{2n+1}$ then $\lim a_n$ does not exist

[1]This assumption is critically important. Talk to me if you're curious why