Math 1B PDP Worksheet

Sequences

1. For each of the following sequences, write the form of the general term a_n, starting your indexing at $n = 1$. Also determine whether each sequence is convergent or divergent. For those that are convergent, find the limit.
   (a) $\{1, 2, 3, 4, \ldots\}$
   (b) $\{2, -2, 2, -2, 2, \ldots\}$
   (c) $\{4, 7, 10, 13, \ldots\}$
   (d) $\{\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\}$
   (e) $\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots\}$
2. Determine whether each of the following sequences are convergent or divergent. For those that are convergent, find the limit.
   (a) $a_n = \frac{3n^2+1}{n^2-1}$
   (b) $a_n = \frac{(n+2)!}{n^2 \cdot n!}$
   (c) $a_n = \cos 2\pi n$
   (d) $\{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \ldots\}$
   (e) $a_n = \ln(n^2 - 3n + 1) - \ln(n^2 + 4)$
   (f) $a_n = \frac{\sin n}{n}$
   (g) $a_n = n \tan(1/n)$
3. Consider the sequence $a_n = r^n$, where r is a constant.
   (a) Write out a few terms of this sequence for $r = -2, -1, 1/2, 1, 2$. What is the limit in each of these cases?
   (b) In general, for what values r does this sequence converge? Find the limit for those values.
   (c) Repeat part (b) for the sequence $a_n = nr^n$.
4. Let $p(x) = b_l x^l + b_{l-1} x^{l-1} + \cdots + b_0, q(x) = c_m x^m + c_{m-1} x^{m-1} + \cdots + c_0$ be polynomials of degrees l, m respectively. Define a sequence a_n by $a_n = \frac{p(n)}{q(n)}$. Determine whether $\lim_{n\to\infty} a_n$ exists in each of the following cases. When it does, find its value.
   (a) $\deg p < \deg q$
   (b) $\deg p = \deg q$
   (c) $\deg p > \deg q$
5. True/False. For all problems, a_n and b_n are sequences. Justify your answers with a sketch of a proof or a counterexample.
   (a) If a_n and b_n converge, then $a_n + b_n$ converges.
   (b) If $a_n + b_n$ converges, then a_n and b_n converge.
   (c) If a_n and b_n converge, then a_n/b_n converges.
   (d) If a_n and b_n diverge, then $a_n + b_n$ diverges.
   (e) If $a_n + b_n$ diverges, then a_n and b_n diverge.
   (f) If a_n and b_n diverge, then $a_n b_n$ diverges.

The $\epsilon - N$ definition of a limit

1. (a) Prove, using the $\epsilon - N$ definition, that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.
   (b) Find a counterexample to the statement "If $\lim\limits_{n\to\infty} |a_n| = L$, then $\lim\limits_{n\to\infty} a_n = L$"
2. Prove, using the $\epsilon - N$ definition, that $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$
3. (hard) Prove, using the $\epsilon - N$ definition, that if $\lim\limits_{n\to\infty} a_n = L, \lim\limits_{n\to\infty} b_n = K$ then $\lim\limits_{n\to\infty} (a_n + b_n) = L + K$