First Order Linear ODEs

1. (a) Show that $xy' = y$ is both separable and linear by putting it into the appropriate form for each method.
   (b) Solve the equation using each of the methods. Do you get the same thing?

2. For each of the following equations, decide whether it is separable, linear, neither or both. **For those that are linear,** find the general solution

   (a) $yy' = x\sqrt{1+x^2}\sqrt{1+y^2}$
   (b) $xy' - 2y = x^3$
   (c) $xy' = y + x\cos^2(y/x)$
   (d) $1 + y^2 - y'\sqrt{1-x^2} = 0$
   (e) $y' = x + y$
   (f) $xy' - \dfrac{y}{x+1} = x; \quad y(1) = 0$
   (g) $1 + 2xy^2 + 2x^2yy' = 0$

3. A large tank initially contains 100L of pure water. At a rate of 3L/min, saltwater containing 2kg of salt per liter is pumped into the tank. The solution is kept thoroughly mixed and solution drains through a drain at the bottom of the tank at 2L/min.

   (a) If $S(t)$ is the number of kilograms of salt in the tank after t minutes, explain why S should be a solution to the initial value problem

$$S' = 6 - 2\frac{S}{100+t}; \quad S(0) = 0$$

   (b) Solve this initial value problem to find an explicit formula for $S(t)$

Extra/Hard Problems If you finish early, take a stab at these.

1. (Bernoulli)

   (a) Show that the substitution $u(x) = y^{1-n}$ will **always** transform $y' + P(x)y = Q(x)y^n$ into a linear equation assuming $n \neq 0$. It may be helpful to start by dividing through by y^n.
   Hint: If $u(x) = y^{1-n}$, what are y' and y in terms of u?
   (b) Use this method to solve the differential equation $y' + xy = xe^{-x^2}y^{-3}$

2. Consider the differential equation $y' + xy = x^3$.

   (a) What is the general solution to this equation?
   (b) Let y_1 and y_2 be any two (distinct) solutions to this differential equation, and let $y_0 = y_1 - y_2$. Is y_0 a solution to this ODE? If yes, prove it. If not, what differential equation does it satisfy?
   (c) Now do the same for $y_0 = cy_1$, where c is any constant.

3. (Make sure you've done 2 first)

   (a) Prove that if y_1 and y_2 are solutions to $y' + p(x)y = 0$, then $y_0 = cy_1 + y_2$ is also a solution to the same equation for any choice of the constant c.
   (b) Show that $1/x$ and 0 both satisfy the differential equation $y' + y^2 = 0$, but that $c/x + 0$ does not unless $c = 0$ or $c = 1$. Why does this not contradict part (a)?

4. Show that if r is a double root of $p(x)$, then r is also a root of $p'(x)$. Hint: r is a double root if and only if $p(x)$ is divisible by $(x-r)^2$