Math 1B PDP Worksheet

Instructions

1. Introduce yourselves!
2. Find some blackboard space, a piece of chalk, and decide who will be your first scribe.
3. Do the problems below, having a different person be the scribe for each one.

Tougher Improper Integrals

1. For which values of p does $\int_e^\infty \frac{dx}{x(\ln x)^p}$ converge?
2. For what values of a does $\int_a^\infty \frac{t+3}{t^3+t^2}dt$ converge?
3. Show that $\int_0^\infty \frac{1}{x^p}dx$ diverges for all p
4. For what values of C does $\int_0^\infty \frac{x}{x^2+1} - \frac{C}{3x+1}dx$ converge? What is the value of the integral for this choice of C?
5. Show that $\int_0^\infty \frac{\ln x}{1+x^2} = 0$ by showing $\int_0^1 \frac{\ln x}{1+x^2}dx = -\int_1^\infty \frac{\ln x}{1+x^2}dx$ **and that both of these converge**[1]. Hint: try the substiution $u = 1/x$.

The Comparison Test

1. Decide what relationship ($\leq, \geq, =$, etc), if any, holds between each of the following pairs of functions on the given intervals:

   (a) $x, x^2; 0 \leq x \leq 1$

   (b) $\frac{1}{x^2+x}, \frac{1}{x^2}; 1 \leq x < \infty$

   (c) $\frac{1}{2x}, \frac{1}{x^2+x}; 0 < x < 1$

   (d) $\frac{1}{x \sin x}, \frac{1}{x}; 0 < x < \pi/2$

   (e) $\frac{\tan^{-1} x}{x^3}, \frac{2}{x^2}; 1 < x < \infty$

2. Without actually evaluating the given integrals, determine whether each of the following converges.

   (a) $\int_1^\infty \frac{\cos^2 x}{x^2+3}$

   (b) $\int_1^\infty \frac{1+\sin x}{2^x+5^x}$

   (c) $\int_0^1 \frac{\tan^{-1} x}{\sqrt{x+x^2}}$

   (d) $\int_3^\infty \frac{1}{x^2-2x^3+x^4}dx$

   (e) $\int_0^1 \frac{x^3-2x^2+3\sqrt{x}}{\sqrt[3]{x^{10}-x^7+4x^3}}dx$

   (f) $\int_1^3 \frac{e^x}{\sqrt{x^2-4}}dx$

Extra Problems If you finish early, take a stab at these.

1. Find each of the following.

   (a) $\int \frac{\cos x - 1}{\cos x + 1}dx$

   (b) $\int \frac{\sin x + \cos x}{\sin 2x}dx$

   (c) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

   (d) $\int \frac{1}{x\sqrt{x-1}}$

2. Let S be a sphere of radius 1. Now take a plane a distance $d < 1$ away from the center and slice the sphere into two pieces. What is the volume of each piece? Hint: the formula for the volume of a revolved solid is $\int_a^b \pi f(x)^2$
3. Consider the integral $\int_0^\infty x^n e^{-x}dx$

   (a) Evaluate this integral for $n = 0, 1, 2, 3$

   (b) Using your answer to (a), guess a formula for the value of this integral for any positive n.

[1] If you don't also show convergence, you haven't proved anything since $\infty - \infty \neq 0$

(c) If you know how to do induction, use it to prove your guess in (b) is correct.

4. $\left(\text{Why we do } \lim_{t\to\infty}\int_0^t + \lim_{s\to-\infty}\int_s^0\right)$

(a) Consider the integral $\int_{-\infty}^{\infty} 2x dx$. Show that this integral diverges.

(b) Evaluate each of the following and explain why from a naive standpoint they seem like $\int_{-\infty}^{\infty}$

i. $\displaystyle\lim_{t\to\infty}\int_{-t}^{t} 2x dx$

ii. $\displaystyle\lim_{t\to\infty}\int_{-t}^{t+1} 2x dx$

iii. $\displaystyle\lim_{t\to\infty}\int_{-t}^{\sqrt{t^2+1}} 2x dx$

(c) Convince yourself that part (b) really does show that $\displaystyle\int_{-\infty}^{\infty} = \lim_{t\to\infty}\int_{-t}^{t}$ is **not** a good definition.