Math 1B PDP Worksheet

Instructions

1. Introduce yourselves!
2. Find some blackboard space, a piece of chalk, and decide who will be your first scribe.
3. Do the problems below, having a different person be the scribe for each one.

Improper Integrals

1. Which of the following integrals are improper? For those that are improper, show how you would break them up into the limit of proper integrals and then decide whether they converge or diverge. Do not use the comparison test.

   (a) $\int_0^{\pi/2} \sec x dx$

   (b) $\int_{-3}^{3} \frac{1}{x^2+1} dx$

   (c) $\int_4^{\infty} e^{-y/2} dy$

   (d) $\int_{-\infty}^{\infty} \frac{1}{x^3} dx$

   (e) $\int_0^1 \frac{dy}{4y-1}$

   (f) $\int_{-1}^{1} \frac{e^x}{e^x-1} dx$

   (g) $\int_0^{\infty} \sin t dt$

2. Consider the integral $\int_{-1}^{1} \frac{1}{x^{4/3}}$.

   (a) Explain why this integral is improper and show it diverges.

   (b) What would you get if you "forgot" that it was improper and just evaluated its anti-derivative at the endpoints? Why does this answer not make any sense?

3. $\left(\text{Why we do } \lim_{t\to\infty} \int_0^t + \lim_{s\to-\infty} \int_s^0\right)$

   (a) Consider the integral $\int_{-\infty}^{\infty} 2x dx$. Show that this integral diverges.

   (b) Evaluate each of the following and explain why from a naive standpoint they seem like $\int_{-\infty}^{\infty}$

      i. $\lim_{t\to\infty} \int_{-t}^{t} 2x dx$

      ii. $\lim_{t\to\infty} \int_{-t}^{t+1} 2x dx$

      iii. $\lim_{t\to\infty} \int_{-t}^{\sqrt{t^2+1}} 2x dx$

   (c) Convince yourself that part (b) really does show that $\int_{-\infty}^{\infty} = \lim_{t\to\infty} \int_{-t}^{t}$ is **not** a good definition.

4. Find $\int_0^{\infty} \frac{\ln x}{1+x^2} dx$

Integration Practice

1. Find each of the following integrals. Be sure to work as a group so **everyone** knows how to do all these problems.

   (a) $\int e^{x+e^x} dx$

   (b) $\int \frac{\sec^2(\sin\theta)}{\sec\theta} d\theta$

   (c) $\int \frac{1}{\sqrt{x+1}+\sqrt{x}} dx$

   (d) $\int \frac{\ln(x+1)}{x^2} dx$

(e) $\int \frac{t^3+1}{t^3-t^2}dt$

(f) $\int \cos^4 t - \sin^4 t dt$

(g) $\int \cos^3 2x \sin 2x dx$

(h) $\int \frac{dt}{\sqrt{e^t}}$

(i) $\int \frac{1}{x\sqrt{x^2+4}}dx$

(j) $\int \frac{1}{x\sqrt{x+4}}dx$

(k) $\int \frac{x}{\sqrt{x^2+4}}dx$

(l) $\int \frac{1}{\sqrt[4]{x}+\sqrt[3]{x}}$

(m) $\int \ln(\sec\theta)\sec^2\theta d\theta$

Extra Problems If you finish early, take a stab at these.

1. (The Weirstrass Substitution) It turns out that **any** rational function of sin and cos (and hence, of $\sec, \tan, \csc, \cot$, etc) can be turned into an ordinary rational function via the substitution $t = \tan(\frac{x}{2})$. Let's explore why:

   (a) Show that $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}}$ and that $\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}}$. Hint: right triangles

   (b) Use trig identities to show that $\cos x = \frac{1-t^2}{1+t^2}$, and that $\sin x = \frac{2t}{1+t^2}$

   (c) Show that $dx = \frac{2}{1+t^2}dt$

   (d) Use parts (b) and (c) to find $\int \frac{dx}{3\sin x - 4\cos x}$ and $\int \sec^3 x dx$

2. Find each of the following. While you could use the technique of the previous problem for (a) and (b), you should try to find a more straightforward way.

   (a) $\int \frac{\cos x - 1}{\cos x + 1}dx$ (b) $\int \frac{\sin x + \cos x}{\sin 2x}dx$ (c) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

3. Let S be a sphere of radius 1. Now take a plane a distance $d < 1$ away from the center and slice the sphere into two pieces. What is the volume of each piece? Hint: the formula for the volume of a revolved solid is $\int_a^b \pi f(x)^2$