Undetermined Coefficients Strategy

Equations of the form $ay'' + by' + cy = G(x)$ with $G(x) \neq 0$ are called non-homogeneous and can take significantly more work to sovle than their homogeneous counterparts. However, when $G(x)$ is of one of the following forms, the Method of Undetermined Coefficients can make this more straightforward. The strategy is as follows:

1. Solve the associated homogeneous equation $ay'' + by' + cy = 0$ by finding the roots of $ar^2 + br + c = 0$

2. Let $y_h =$
   - $C_1 e^{r_1 x} + C_2 e^{r_2 x}$ if there are two real roots r_1, r_2
   - $C_1 e^{rx} + C_2 x e^{rx}$ if there is just one real root
   - $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ if $r = \alpha \pm \beta i$

3. Figure out the form of y_p as follows based on $G(x)$:

   (a) $\underline{e^{kx} P_n(x)}$ where $P_n(x)$ is a degree n polynomial:
   - If k is not a root of $ar^2 + br + c = 0$, then
   $$y_p = e^{kx} Q_n(x) \text{ where } Q_n(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$$
   - If k is a single root, then $y_p = x e^{kx} Q_n(x)$
   - If k is a double root, then $y_p = x^2 e^{kx} Q_n(x)$

   (b) $\underline{e^{\gamma x} \left(P_n(x) \cos \delta x + S_m(x) \sin \delta x\right)}$ where P_n and S_m are polynomials of degrees n, m respectively
   - If $\gamma + \delta i$ is not a root of $ar^2 + br + c = 0$, then
   $$y_p = e^{kx}(Q_N(x) \cos \delta x + R_N(x) \sin \delta x) \text{ where } \begin{cases} N = \max(n, m) \\ Q_N(x) = A_N x^N + A_{N-1} x^{n-1} + \cdots + A_1 x + A_0 \\ R_N(x) = B_N x^N + B_{N-1} x^{N-1} + \cdots + B_1 x + B_0 \end{cases}$$
   - If $\gamma + \delta i$ is a root, then $y_p = x e^{kx}(Q_N(x) \cos \delta x + R_N(x) \sin \delta x)$

   (c) $\underline{G(x) \text{ is a sum of terms}}$ that look like the above types.
   In this case, y_p is just the sum of what you whould have used for each term separately.

4. Solve for the undetermined coefficients in your y_p (the A, B, A_n, B_0, etc). To do this, calculate $ay_p'' + by_p' + cy_p$, set it equal to $G(x)$ and equate coefficients of like terms.

5. Then the general solution is $y = y_h + y_p$

6. If it's an initial/boundary value problem, plug in appropriate values and solve for C_1 and C_2. Note: you **must** include the y_p part when doing this step.

Full Example: Find the general solution to $y'' - 4y = e^{2x} + x \cos x$

1. $r^2 - 4 = 0$ has solutions $r = 2, -2$

2. $y_h = C_1 e^{2x} + C_2 e^{-2x}$

3. This is a sum of e^{2x} and $x \cos x$ terms, so we'll handle them separately for the moment:
   - e^{2x}: Here we're in case (a) with $k = 2, n = 0$. Since 2 is a root, we use $x(Ae^{2x})$
   - $x \cos x$: This is case (b) with $\gamma = 0, \delta = 1, n = 1, m = 0$. Since $0 + i$ is not a root, we use $e^{0x}((A_1 x + A_0) \cos x + (B_1 x + B_0) \sin x)$

   We now combine these (and re-label to avoid confusion) to get
   $$y_p = Axe^{2x} + (Bx + C) \cos x + (Dx + E) \sin x$$

4. We now need to solve for A, B, C, D, E:
   $$\begin{aligned} y_p' &= Ae^{2x} + 2Axe^{2x} + B \cos x - (Bx + C) \sin x + D \sin x + (Dx + E) \cos x \\ y_p'' &= 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} - B \sin x - (B \sin x + (Bx + C) \cos x) + D \cos x + D \cos x - (Dx + E) \sin x \\ &= 4Ae^{2x} + 4Axe^{2x} + (-Bx - C + 2D) \cos x + (-Dx - 2B - E) \sin x \end{aligned}$$

$$
\begin{aligned}
y_p'' - 4y_p &= 4Ae^{2x} + 4Axe^{2x} + (-Bx - C + 2D)\cos x + (-Dx - 2B - E)\sin x \\
&\quad - 4(Axe^{2x} + (Bx + C)\cos x + (Dx + E)\sin x) \\
&= 4Ae^{2x} + (-5Bx - 5C + 2D)\cos x + (-5Dx - 2B - 5E)\sin x
\end{aligned}
$$

We want this to equal $e^{2x} + x\cos x$, so we need to solve the system of equations

$$
\begin{cases}
4A &= 1 \\
-5B &= 1 \\
-5C + 2D &= 0 \\
-5D &= 0 \\
-2B - 5E &= 0
\end{cases}
$$

We can quickly see that $A = \frac{1}{4}, B = -\frac{1}{5}, D = 0$. From here, we can plug into the third equation to get $-5C + 0 = 0$, so $C = 0$. Plugging into the last equation give $\frac{2}{5} - 5E = 0$, so $E = \frac{2}{25}$. Putting it all together we get:

$$
y_p = \frac{1}{4}xe^{2x} - \frac{1}{5}x\cos x + \frac{2}{25}\sin x
$$

5. Combining everything, we see that

$$
y = C_1e^{2x} + C_2e^{-2x} + \frac{1}{4}xe^{2x} - \frac{1}{5}x\cos x + \frac{2}{25}\sin x
$$

is the general solution

Finding y_p examples: Here's some examples of just step 3.

- $y'' - 2y' + 5y = x^2e^x$

  This is an example of case (a) with $n = 2, k = 1$. $r^2 - 2r + 5 = 0$ has solutions $1 \pm 2i$, so k is not a root an thus we use
  $$y_p = (Ax^2 + Bx + C)e^x$$

- $y'' - 2y' + 5y = e^x\cos 2x + xe^x\sin 2x$

  This is an example of case (b), with $\gamma = 1, \delta = 2, n = 0, m = 1$. Since $r^2 - 2r + 5 = 0$ has solutions $r = 1 \pm 2i$, $\gamma + \delta i$ is a root and thus we use
  $$y_p = xe^x\left((Ax + B)\cos 2x + (Cx + D)\sin 2x\right)$$

- $y'' + 2y' + y = (x^2 + 2)e^{-x}$

  This is an example of case (a) with $k = -1, n = 2$. Since -1 is a double root of $r^2 + 2r + 1$, we use
  $$y_p = x^2(Ax^2 + Bx + C)e^{-x}$$

- $y'' + 4y = \cos 2x$

  This is an example of case (b) with $\gamma = 0, \delta = 2, n = 0, m = 0$. Since $2i$ is a root of $r^2 + 4 = 0$, we use
  $$y_p = x(A\cos x + B\sin x)$$

- $y'' - 2y' + 5y = x^2e^x + e^x(\cos 2x + x\sin 2x)$

  This is the sum of the first two examples above, so we would use the same logic on each piece separately to get
  $$y_p = (Ax^2 + Bx + C)e^x + xe^x\left((Dx + E)\cos 2x + (Fx + G)\sin 2x\right)$$

- $y'' + 9y = \cos x + \sin 3x$ There are two separate parts here: the $\cos x$ part for which we use $A\cos x + B\sin x$ and the $\sin 3x$ part for which we note that $3i$ is a root of $r^2 + 9$ and thus use $x(C\cos 3x + D\sin 3x)$. We therefore use
  $$y_p = A\cos x + B\sin x + x(C\cos 3x + D\sin 3x)$$