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Math 1B: A Summary[1]

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Integration

- u-Substitution
  - Good choices for u are things whose derivative (ie, du) is somewhere else in the problem
  - Square roots or things inside square roots can sometimes also be good choices
  - Random tricks:
    - Can solve your $u = ...$ equation for x if you need to deal with x terms that aren't part of your u or du
    - A square root in the denominator becomes part of du if you use it for your u
  - Remember to plug back in afterwards!
- By Parts
  - $\int u dv = uv - \int v du$
  - To pick u, use LIATE (Logs, Inverse Trig, Algebra (ie polynomials), Trig, Exponentials). dv is everything else
  - Sometimes (esp. with logs or inverse trig) setting u =everything, $dv = dx$ can work
- Trig Identities
  - $\sin^m \cos^n$
    - If one is odd, save one and convert the rest
    - If both even, use double/half angle ID's
  - $\sec^m \tan^n$
    - If sec is even, save two and convert the rest to tan
    - If tan is odd, save a $\sec x \tan x$ and convert everthing to sec
    - Otherwise, use identities
  - Other trig functions
    - Use trig identities to either simplify to things you know how to integrate, or to setup a u-sub
- Trig Substitutions
  - $\sqrt{x^2 - a^2} \Rightarrow x = a \sec\theta, 0 \leq \theta < \frac{\pi}{2}, \pi \leq \theta < \frac{3\pi}{2}$
  - $\sqrt{a^2 - x^2} \Rightarrow x = a \sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
  - $\sqrt{x^2 + a^2} \Rightarrow x = a \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
  - For quadratics under square roots, try completing the square and making a trig sub
  - Don't forget to replace dx too!
  - To plug back in, draw a triangle and use SohCahToa or use trig IDs
- Partial Fractions
  - Only works for $\frac{\text{polynomial}}{\text{polynomial}}$
  - Always start with long division to make deg(num) < deg(denom)
  - Factor denominator and find form of decomposition:
    - Distinct linear factors: $\frac{A}{ax+b}$
    - Repeated linear factors: $\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + ...$
    - Distinct quadratic factors: $\frac{Ax+B}{ax^2+bx+c}$
    - Repeated quadratic factors: $\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + ...$
  - Cross multiply and solve for coefficients by equating powers of x
  - Integrate term-by-term

Improper Integrals

- Just Integrate
  - Turn into a limit, find an antiderivative (see above), and take limit
  - Most often used to determine behavior of g in CT/LCT
- p-integrals
  - $\displaystyle\int_0^a \frac{1}{x^p}$ conv. for $p < 1$, div. for $p \geq 1$
  - $\displaystyle\int_a^\infty \frac{1}{x^p}$ conv. for $p > 1$, div. for $p \leq 1$
- Comparison Test
  - Only works for positive integrands
  - If $f(x) \leq g(x)$ and $\int_a^b g$ conv., then $\int_a^b f$ conv.
  - If $f(x) \geq g(x)$ and $\int_a^b g$ div., then $\int_a^b f$ div.
  - **Never** introduce/move impropernesses when picking g
- Limit Comparison Test
  - Only works for positive integrands
  - To pick $g(x)$, factor f as much as possible and then plug in "improperness" into each term. You care about 0's and ∞'s
  - Use Basic Limit Laws for $\sin(0), \ln(1)$, etc
  - **Never** introduce/move impropernesses when picking g

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[1] As always, I make no guarantees this is complete or accurate. Let me know if you find any errors/omissions. To save space I omit many things (dx in integrals, "(x)" in $f(x)$, etc) that you should not. Do as I say, not as I do. Also, this intentionally omits topics, such as Surface Area and Applications, which will not be on the final.

1st Order Differential Equations (ie, y' but no y'')

- Separable
  - Re-write as $f(y)dy = g(x)dx$. If you divide through by any y terms, check for constant solutions
  - Integrate both sides. Remember your $+C$
  - Solve for y
  - If you divided by some y term, check if $(\textit{that term}) = 0$ gives any constant solutions
- Linear
  - Re-write as $y' + P(x)y = Q(x)$
  - Compute $I = Ae^{\int P(x)}$. By choosing appropriate A, can drop abs. value bars
  - $y = \frac{\int IQ}{I}$. Don't forget your $+C$ in the numerator.

Sequences[2]

- Mostly tricks from 1A
- L'Hospital's rule
  - Make sure to re-write using x first
- Multiply top and bottom by conjugate
- Take log and then e your answer
  - Use when n in both base and exponent
- Subsequences
  - If even/odd subsequences have different limits, original sequence diverges
  - If even/odd subsequences have same limit, then original sequence has that limit too
  - Especially useful for things with a $(-1)^n$
- Absolute value
  - If $\lim |a_n| = 0$, then $\lim a_n = 0$
  - **No information** if $\lim |a_n| \neq 0$
- Squeeze Theorem
  - If $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$
  - **No information** if $\lim a_n \neq \lim c_n$

Series $\left(\text{ie}, \sum_{n=?}^{\infty} a_n\right)$

- Geometric Series
  - $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise
- Test for Divergence
  - If $\lim a_n \neq 0$ (see sequences section), then $\sum a_n$ diverges
- Integral Test
  - Only works for positive, continuous, eventually decreasing functions/series
- Comparison Test
  - Only works for series with only positive terms
  - If $a_n \leq b_n$ (eventually) and $\sum b_n$ conv., then $\sum a_n$ conv.
  - If $a_n \geq b_n$ (eventually) and $\sum b_n$ div., then $\sum a_n$ div.
- Limit Comparison Test
  - Only works for series with only positive terms
  - If $\lim \frac{a_n}{b_n} \neq 0, \infty$ (and exists), then $\sum a_n$ and $\sum b_n$ behave the same
  - If $\lim \frac{a_n}{b_n} = 0$ and $\sum b_n$ conv., then $\sum a_n$ conv.
  - If $\lim \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div., then $\sum a_n$ div.
- Alternating Series Test
  - $\sum (-1)^n b_n$ converges if b_n is decreasing and $\lim b_n = 0$
- Absolute Convergence
  - If $\sum |a_n|$ conv, then $\sum a_n$ is absolutely convergent (and thus conv)
  - Useful if you want to use the CT/LCT/IntTest and series has both + and - terms
- Ratio Test
  - $\sum a_n$ absolutely convergent if $\lim |\frac{a_{n+1}}{a_n}| < 1$
  - Divergent if > 1
  - **No information** if $= 1$
- Root Test
  - $\sum a_n$ absolutely convergent if $\lim \sqrt[n]{|a_n|} < 1$
  - Divergent if > 1
  - **No information** if $= 1$

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[2] While you won't have any explicit sequence problems, they could show up as part of a test for divergence or AST.

Power Series

- Finding ROC and IOC
  1. Start with root or ratio test, evaluate limit, and set less than 1
  2. Re-write in form $|x-a| < R$ (Note: $|2x-3| < 3$ is **not** in this form because of the 2)
  3. Endpoints are $a-R$ and $a+R$. Plug in and use other tests to determine behavior at endpoints
- Finding a Macluarin series
  - Figure out which known series to start with and write it down
  - Apply operations to both sides of equation to make the "function" side what you want
  - To find IOC, plug in any substitutions you made into interval for known series. Need to recheck endpoints if you took derivative or integral.
  - Reminder: if you integrate, don't forget to sovle for the $+C$
- Finding a Taylor Series
  - Replace x by $x-a$ in given function and add/subtract to compensate
  - Follow same steps as above
- Finding $f^{(m)}(a)$
  - Find Taylor Series centered at a. Factor out if necessary to make it $\sum a_n(x-a)^{something}$
  - Find which n gives m as exponent of x Use this to find c_m
  - $f^{(m)}(a) = c_m m!$
- Finding the sum of a series
  - Look at denominator to figure out which known series to start from
  - If no "x", replace numbers by x's so you can plug in later
  - Apply operations to both sides of known series until "series" side matches problem
  - Answer is the "function" side. Plug in your x value if necessary
  - See previous handout

2nd Order Differential Equations (ie, something including y'')

- Homogeneous ($ay'' + by' + cy = 0$)
  1. Find roots of $ar^2 + br + c = 0$
  2. Find general solution
     - Two real roots: $y = C_1e^{r_1x} + C_2e^{r_2x}$
     - One real root: $y = C_1e^{rx} + C_2xe^{rx}$
     - One pair of complex roots: $y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$
- Undetermined Coefficients ($ay'' + by' + cy = G(x)$)
  1. Use method above to find roots and y_c
  2. Determine form of y_p as follows:
     - $G(x) = P_n(x)e^{kx}$
       - k not a root: $y_p = (A_nx^n + \cdots + A_0)e^{kx}$
       - k single root: $y_p = x(A_nx^n + \cdots + A_0)e^{kx}$
       - k dobule root: $y_p = x^2(A_nx^n + \cdots + A_0)e^{kx}$
     - $G(x) = e^{\gamma x}P_n(x)\cos\delta x + e^{\gamma x}Q_n(x)\sin\delta x$
       - $\gamma + \delta i$ not a root: Let $N = \max(n, m)$ $y_p = e^{\gamma x}(A_Nx^N + \cdots + A_0)\cos\delta x + e^{\gamma x}(B_Nx^N + \cdots + B_0)\sin\delta x$
       - $\gamma + \delta i$ is a root: $y_p = x(\text{above})$
     - Sum of these
       - Sum of guesses for each summand
  3. Plug into $ay'' + by' + cy$, group like terms, and set $= G(x)$
  4. Solve for A, B, C, etc by equating like terms on each side
  5. $y = y_c + y_p$ (Note: your only remaining unknown coefficients should be C_1 and C_2)
- Variation of Parameters ($ay'' + by' + cy = G(x)$)
  - Mostly used when $G(x)$ isn't one of the three "Types" for undetermined coefficients
  - Find y_1 and y_2 by finding y_c as above
  - Solve system of equations $\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = G(x) \end{cases}$ for u_1', u_2'
  - Integrate to get u_1, u_2. Remember your $+C_1$ and $+C_2$
  - $y = u_1y_1 + u_2y_2$
- Initial/Boundary Value Problems
  - Use appropriate method from above to find general solution, then plug in to solve for C_1 and C_2