Math 32 Final Exam

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You have until 3:30pm to complete this test. No calculators, books, notes, or consultation with other members of the class are permitted. Your exam should have 18 pages, the last of which is a list of trig identites that you may tear off if you wish.

Unsupported or improperly supported answers will receive no credit.

Name: ______________________________

GSI: ______________________________

Section time: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	15	
6	10	
7	10	
8	15	
9	15	
10	10	
11	10	
12	10	
13	15	
Total:	150	

1. Short answer. You need not show any work for this problem

   (a) (4 points) Evaluate each of the following:

   - $\ln(\sqrt{e}) + e^{\ln 3 - \ln 4} =$

     Solution: $\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$

   - $\cos\left(\frac{4\pi}{3}\right) =$

     Solution: $-\frac{1}{2}$

   - $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) =$

     Solution: $-\frac{\pi}{3}$

   - $\sin^{-1}\left(\sin\left(\frac{23\pi}{5}\right)\right) =$

     Solution: $\frac{23\pi}{5} - 4\pi = \frac{3\pi}{5}$

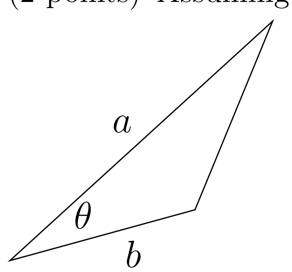
   (b) (2 points) Find an equation for the line passing through $(2, 3)$ and $(-1, -3)$.

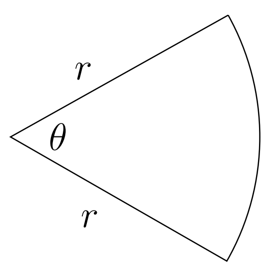
   Solution: $m = \frac{3-(-3)}{2-(-1)} = 2$
   $y - 3 = 2(x - 2)$

   (c) (2 points) Find a quadratic polynomial with real coefficients that has $2 + i$ has a root

   Solution: $(x - (2+i))(x - (2-i)) = x^2 - (2+i)x - (2-i)x + (2+i)(2-i) = x^2 - 4x + 5$

   (d) (2 points) Assuming θ is measured in radians, find the area of each figure:





$A =$ $\qquad\qquad$ $A =$

Solution: $\frac{1}{2}ab\sin\theta$ $\qquad\qquad$ $\frac{1}{2}r^2\theta$

2. (10 points) Find all solutions to $\log_4 x = \log_2(x-2)$ Hint: convert the LHS to base 2

Solution:

$$\begin{aligned}
\log_4 x &= \log_2(x-2) \\
\frac{\log_2 x}{\log_2 4} &= \log_2(x-2) \\
\frac{\log_2 x}{2} &= \log_2(x-2) \\
\sqrt{x} &= x-2 \\
x &= x^2 - 4x + 4 \text{ (*Check work later)} \\
0 &= x^2 - 5x + 4 \\
0 &= (x-4)(x-1)
\end{aligned}$$

So the potential solutions are $x = 4$ and $x = 1$.

Plugging $x = 1$ into the original equation gives a $\log_2(-1)$ term, so it can't be a solution since log isn't defined for negative numbers.

Plugging $x = 4$ in gives:

$$\begin{aligned}
LHS &= \log_4 4 \\
&= 1 \\
RHS &= \log_2(4-2) \\
&= \log_2 2 = 1
\end{aligned}$$

So it is a solution. Thus, the only solution is $x = 1$

3. (10 points) Sketch a graph of $y = \dfrac{(x-3)(x+1)}{x(x-1)}$. Be sure to label the coordinates of any important features, including intercepts, asymptotes, and places where the graph crosses an asymptote (if any).

Solution:

Since this is already factored, we can quickly see that the vertical asymptotes are $x = 0$ and $x = 1$ and that the x-intercepts are $x = 3$ and $x = -1$. Because 0 is not in the domain, there is no y-intercept.

The degree of the numerator and denominator are both 2, so the horizontal asymptote is just the quotient of the leading coefficients and thus is 1. To find where (if ever) the graph crosses its asymptote we solve:

$$\begin{aligned} \frac{(x-3)(x+1)}{x(x-1)} &= 1 \\ (x-3)(x+1) &= x(x-1) \quad \text{(*Could introduce extraneous solutions x=0, x=1)} \\ x^2 - 2x - 3 &= x^2 - x \\ -3 &= x \end{aligned}$$

So it crosses at $x = -3$.

Finally we note that between $x = 0$ and $x = 1$ the function is positive.

Combining all this gives:

4. (10 points) Find the domain of the function $g(x) = \dfrac{\sin^{-1}(x/2)}{\ln(1+x) - 1}$. Write your answer using interval notation.

Solution: There are 3 potential problems:

- The domain of $\sin^{-1} t$ is $[-1, 1]$, so we need $-1 \leq \frac{x}{2} \leq 1$. Multiplying through by 2 gives $-2 \leq x \leq 2$
- The domain of $\ln t$ is $(0, \infty)$, so we need $1 + x > 0$, which is the same as saying $x > -1$
- The denominator can't be zero, so we need to avoid any x's that would make $\ln(1+x) = 1$. Exponentiating both sides gives $1 + x = e$, so we need to avoid $x = e - 1$ (Note that $e - 1 \approx 1.71$ so neither of the above cases eliminated it).

Combining all this gives that the domain is $(-1, e-1) \cup (e-1, 2]$

5. (a) (5 points) Let $f(x) = \sqrt{x}, g(x) = x + 1, h(x) = \frac{1}{x}$. Write $\frac{1}{\sqrt{x+1}} + 1$ as a composition of f, g, h. (ie, your answer should be something like $f \circ h \circ g \circ h$)

Solution:

$$\begin{aligned}\frac{1}{\sqrt{x+1}} + 1 &= g(\frac{1}{\sqrt{x+1}}) \\ &= g(h(\sqrt{x+1}) \\ &= g(h(f(x+1))) \\ &= g \circ h \circ f \circ g\end{aligned}$$

(b) (10 points) Find the inverse of the function $f(x) = 3 - \ln(x+2) + \ln(x-1)$ or explain why no such inverse can exist.

Solution: We do the usual trick of swapping x and y and solving for y:

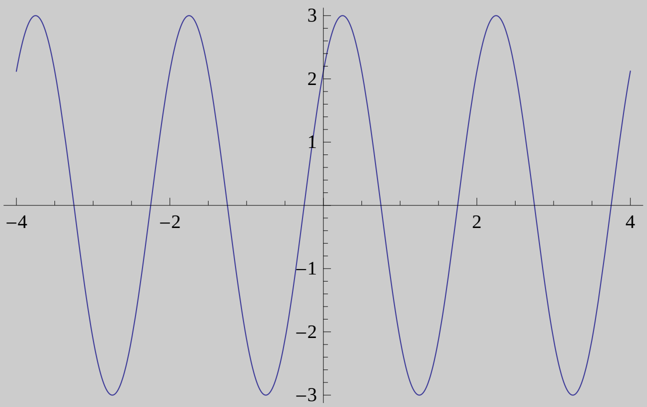
$$\begin{aligned}x &= 3 - \ln(y+2) + \ln(y-1) \\ x &= 3 + \ln\left(\frac{y-1}{y+2}\right) \\ x - 3 &= \ln\left(\frac{y-1}{y+2}\right) \\ e^{x-3} &= \frac{y-1}{y+2} \\ e^{x-3}(y+2) &= y - 1 \\ e^{x-3}y + 2e^{x-3} + 1 &= y \\ 2e^{x-3} + 1 &= y - e^{x-3}y \\ 2e^{x-3} + 1 &= y(1 - e^{x-3}) \\ y &= \frac{2e^{x-3}+1}{1-e^{x-3}}\end{aligned}$$

Since this is a function (no $\pm$ or other things that would make it fail the "vertical line test"), we get:

$$f^{-1}(x) = \frac{2e^{x-3}+1}{1-e^{x-3}}$$

6. (10 points) Sketch a graph of $y = 3\cos(\pi x - \frac{\pi}{4})$. Be sure to label the coordinates of the y-intercept as well as one full period's worth of x-intercepts, peaks, and valleys.

Solution: The amplittute is 3, the period is $2\pi/\pi = 2$ and the phase shift is $\frac{1}{4}$ to the right. The y-intercept is $3\cos(-\pi/4) = 3\sqrt{2}/2$. The x-intercepts for one period of $y = \cos x$ are $\pi/2, 3\pi/2$, so for our function they become $1/2 + 1/4 = 3/4$ and $3/2 + 1/4 = 7/4$. The peaks used to be at $0, 2\pi$ so they're now at $1/4, 9/4$. The valley used to be at π so now it's at $5/4$. Combining all this gives:



7. (10 points) Solve the inequality $|x-1|+|x-2| \geq 3$. Write your answer in interval notation. Hint: consider the three cases $x<1,\ 1 \leq x<2,\ 2 \leq x$

Solution: We'll use the method described in the hint, making use of the definition

$$|y| = \begin{cases} y & y \geq 0 \\ -y & y < 0 \end{cases}$$

- If $x<1$, then $x-1$ and $x-2$ are both negative, so we're in the second case for both absolute values and we get:

$$\begin{aligned} -(x-1)-(x-2) &\geq 3 \\ -x+1-x+2 &\geq 3 \\ -2x &\geq 0 \\ x &\leq 0 \end{aligned}$$

- $if 1 \leq x<2$, then $x-1$ is positive and $x-2$ is negative so what we're really solving is:

$$\begin{aligned} (x-1)-(x-2) &\geq 3 \\ x-1-x+2 &\geq 3 \\ 1 &\geq 3 \end{aligned}$$

  Which obviously has no solutions.

- Finally, we consider the case $x>2$ wherein both $x-1$ and $x-2$ are positive so we just need to solve:

$$\begin{aligned} x-1+x-2 &\geq 3 \\ 2x-3 &\geq 3 \\ 2x &\geq 6 \\ x &\geq 3 \end{aligned}$$

Combining all this, we get $(-\infty, 0] \cup [3, \infty)$ as the full solution

8. Rewrite each of the following expressions so that no trigonometric or inverse trigonometric functions occur.

(a) (5 points) $\tan(\cos^{-1}\frac{x}{2})$

Solution: We draw a right triangle with adjacent = x, hypotenuse = 2. Then the opposite side must be $\sqrt{4-x^2}$ so we have:

$$\tan\left(\cos^{-1}\frac{x}{2}\right) = \frac{\sqrt{4-x^2}}{x}$$

(b) (10 points) $\sin\left(\cos^{-1}(3x) + \tan^{-1}(x+1)\right)$

Solution: Here we'll have to use the $\sin(A+B)$ trig identity first:
To save some notation, let $\theta = \cos^{-1}(3x), \phi = \tan^{-1}(x+1)$

$$\begin{aligned}\sin\left(\cos^{-1}(3x) + \tan^{-1}(x+1)\right) &= \sin(\theta+\phi)\\ &= \sin\theta\cos\phi + \cos\theta\sin\phi\end{aligned}$$

To find $\sin\theta, \cos\theta$, we draw a right triangle with adjacent side $3x$ and hypotenuse 1. Then the opposite side must be $\sqrt{1-9x^2}$ so $\sin\theta = \frac{\sqrt{1-9x^2}}{1} = \sqrt{1-9x^2}$ and $\cos\theta = 3x$.
Similarly, for $\sin\phi, \cos\phi$ we draw a triangle with opposite side $x+1$ and adjacent side 1. Then the hypotenuse is $\sqrt{(x+1)^2+1^2}$. Thus, $\sin\phi = \frac{x+1}{\sqrt{(x+1)^2+1}}$ and $\cos\phi = \frac{1}{\sqrt{(x+1)^2+1}}$.
Plugging all this back in to the equation above gives:

$$\begin{aligned}\sin\left(\cos^{-1}(3x) + \tan^{-1}(x+1)\right) &= \sin\theta\cos\phi + \cos\theta\sin\phi\\ &= \sqrt{1-9x^2}\frac{1}{\sqrt{(x+1)^2+1}} + 3x\frac{x+1}{\sqrt{(x+1)^2+1}}\\ &= \frac{\sqrt{1-9x^2}+3x(x+1)}{\sqrt{(x+1)^2+1}}\end{aligned}$$

9. Let C_1 be the circle whose equation is given by $x^2 - 6x + y^2 + 4 = 0$

(a) (5 points) What are the radius and center of this circle?

Solution: We complete the square:

$$\begin{aligned} x^2 - 6x + y^2 + 4 &= 0 \\ (x-3)^2 - 9 + y^2 + 4 &= 0 \\ (x-3)^2 + y^2 &= 5 \end{aligned}$$

Thus the center is $(3, 0)$ and the radius is $\sqrt{5}$

(b) (10 points) Now let C_2 be the circle of radius $\sqrt{10}$ centered at $(-2, 0)$. Find all places where C_1 and C_2 intersect.
Hint: set up a system of equations

Solution:
The equation for C_2 is $(x+2)^2 + y^2 = 10$, so we just need to solve the system of equations

$$\begin{cases} (x-3)^2 + y^2 = 5 \\ (x+2)^2 + y^2 = 10 \end{cases}$$

Solving the first equation for y^2 gives $y^2 = 5 - (x-3)^2$. Plugging this into the second gives:

$$\begin{aligned} (x+2)^2 + 5 - (x-3)^2 &= 10 \\ x^2 + 4x + 4 + 5 - (x^2 - 6x + 9) &= 10 \\ 10x &= 10 \\ x &= 1 \end{aligned}$$

To get the corresponding $y's$, we just plug in and see $y^2 = 5 - (-2)^2 = 1$, so $y = \pm 1$.
Thus, the solutions are $(1, 1)$ and $(1, -1)$

10. (10 points) Find all solutions to $2\cos^2(4x) = -7\sin(4x) + 5$

Solution: We start by changing the $\cos^2(4x)$ to $1 - \sin^2(4x)$:

$$\begin{aligned} 2(1 - \sin^2(4x)) &= -7\sin(4x) + 5 \\ 0 &= 2\sin^2(4x) - 7\sin(4x) + 3 \end{aligned}$$

So by the quadratic equation, we get $\sin 4x = \dfrac{7 \pm \sqrt{49 - 24}}{4} = \dfrac{1}{2}, 3$

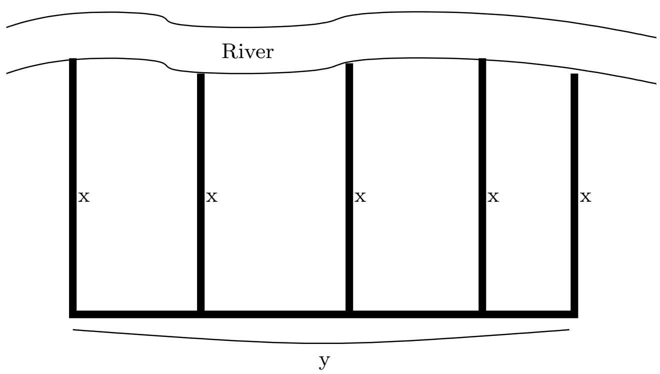
Since $\sin 4x = 3$ has no solutions, we don't need to worry about it. For $\sin 4x = \frac{1}{2}$, we get:

$$4x = \frac{\pi}{6} + 2\pi k, \pi - \frac{\pi}{6} + 2\pi k$$

Simplifying a bit and dividing through by 4 gives the final answer of

$$x = \frac{\pi}{24} + \frac{\pi k}{2},\ x = \frac{5\pi}{24} + \frac{\pi k}{2}$$

11. (10 points) A farmer has a total of 500ft of fencing with which to build four pastures on the bank of a river. Assuming he builds them in the configuration shown in the figure below, what should x and y be so as to maximize the total enclosed area? Note that the river serves as one side of the pasture and doesn't need fencing.



Solution: Our constraint is $5x + y = 500$ and we want to maximize $A = xy$. Solving the constraint for y and plugging in gives

$$A = x(500 - 5x) = -5x^2 + 500x$$

(Sanity check: the leading coefficient is negative, so the parabola opens down and thus we can in fact maximize it)

This is maximized at $x = \frac{-b}{2a} = \frac{-500}{-10} = 50$. The corresponding y is $500 - 5 \cdot 50 = 250$.

Thus, the dimensions should be $x = 50, y = 250$

12. Show that each of the following are identities:

(a) (5 points) $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution: We'll start with the RHS:

$$\begin{aligned}\frac{1-\tan^2\theta}{1+\tan^2\theta} &= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\sec^2\theta}\\ &= \cos^2\theta\left(\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}\right)\\ &= \cos^2\theta\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}\\ &= \cos 2\theta = LHS\end{aligned}$$

(b) (5 points) $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$

Solution: We start by making the (seemingly obvious) observation that $3x = 2x + x$.

$$\begin{aligned}LHS &= \frac{\sin(2x+x)}{\sin x} - \frac{\cos(2x+x)}{\cos x}\\ &= \frac{\sin(2x)\cos x + \cos(2x)\sin x}{\sin x} - \frac{\cos(2x)\cos x - \sin(2x)\sin x}{\cos x}\\ &= \frac{\sin(2x)\cos x}{\sin x} + \cos(2x) - \cos(2x) + \frac{\sin(2x)\sin x}{\cos x}\\ &= \frac{\sin(2x)\cos^2 x + \sin(2x)\sin^2 x}{\sin x\cos x}\\ &= \frac{2\sin x\cos x(\cos^2 x + \sin^2 x)}{\sin x\cos x}\\ &= 2 = RHS\end{aligned}$$

13. (a) (5 points) Compute $f(7)$ where $f(x) = x^6 + 3x^5 - 65x^4 - 37x^3 + 15x^2 - 9x + 16$

Solution:
From the remainder theorem, we know that $f(7)$ will be the same as the remainder when we divide $f(x)$ by $x - 7$:

7	1	3	-65	-37	15	-9	16
		7	70	35	-14	7	14
	1	10	5	-2	1	-2	2

So $f(7) = 2$

(b) (10 points) Find all roots of the polynomial $f(x) = 3x^5 - 11x^4 + 17x^3 - 11x^2 + 2$
Hint: one of the rational roots is a double root

Solution: From the rational roots theorem, we know that the only possible rational roots are $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

1	3	-11	17	-11	0	2
		3	-8	9	-2	-2
	3	-8	9	-2	-2	0

So $x = 1$ is a root. Let's see if it's a double root:

1	3	-8	9	-2	-2
		3	-5	4	2
	3	-5	4	2	0

So it is in fact a double root. Let's try $x = -1$:

-1	3	-5	4	2
		-3	8	-12
	3	-8	12	10

So $x = -1$ is not a root.

Let's try $x = 1/3$:

1/3	3	-5	4	2
		1	-4/3	8/9
	3	-4	8/3	26/9

So it's not a root. Let's try $x = -1/3$:

$-1/3$	3	-5	4	2
		-1	2	-2
	3	-6	6	0

So it is a root and the original polynomial factors as

$$3(x-1)^2(x+\frac{1}{3})(x^2-2x+2)$$

To find the two remaining roots, we just use the quadratic formula: $x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i}{2} = 1 \pm i$

Thus, the roots are

$$x = 1, \frac{1}{3}, 1 \pm i$$

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Trig Identities

Addition Rules:

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Product Rules:

$\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

Sum-To-Product:

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

Double Angle:

$\sin x\cos x = \frac{1}{2}\sin(2x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$\cos(2x) = 1 - 2\sin^2(x)$

$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

$\cos(2x) = 2\cos^2(x) - 1$

$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

Half Angle:

$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$