Logarithms as Integrals

Note: For this section, you **must** use the $\ln x := \int_1^x \frac{1}{t} dt$ definition of the logarithm. You may use whatever properties of integrals you know, but you may **not** use any log laws until you have proved them.

1. (Defining e)
   (a) Show that $\ln 1 = 0$
   (b) By using the right endpoint rule with 3 approximating rectangles, show geometrically that $\ln 4 > 1$
   (c) Use the Intermediate Value Theorem to show that $\ln x = 1$ has a solution.
   (d) Use the Mean Value Theorem to show that this solution is unique. What name do we usually give it?

   Note: We have now defined e to be the unique solution to $\ln x = 1$. You may use that in the following problems, but nothing more. In particular, you may not use any properties of e^x until you have proved them to be true (which won't happen until problem 5)

2. (Defining e^x)
   (a) What must be true of any function in order to guarantee it has an inverse?
   (b) Prove that $\ln x$ has this property. Again, remember that you **must** use the integral definition of $\ln x$
   (c) If we define $\exp(x)$ to be the inverse of $\ln x$, what is $\exp(0)$? $\exp(1)$?
   (d) Sketch a graph of $\exp(x)$ based on the fact that is the inverse of $\ln x$

   Note: we have now defined $\exp(x)$ to be the inverse of $\ln x$.

3. Write a^x in terms of exp and ln.

   Note: this is the only reasonable way to actually define a^x for non-rational a and x. (for example, try to figure out what $\sqrt{2}^\pi$ should mean in terms of "repeated multiplication" or any other "definition" of exponentiation you know)

4. (Log Laws)
   (a) Explain (using the integral definition of $\ln x$) why $\ln(ab) = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$
   (b) By making the substitution $u = t/a$ in the second integral, show that $\ln(ab) = \ln a + \ln b$
   (c) Use the subsitution $u = \sqrt[n]{t}$ to show that $\ln x^n = n \ln x$. Hint: $\frac{d}{dt} \sqrt[n]{t} = \frac{\sqrt[n]{t}}{nt}$

5. (Exp Laws) Use the previous problem to prove that $\exp(x + y) = \exp(x)\exp(y)$ and that $\exp(xy) = (\exp(x))^y$

6. ($\exp(x) = e^x$)
   (a) Justify each equality: $\ln(\exp(x)) = x = x \ln e = \ln e^x$
   (b) Without say "take exp of both sides," explain why $\ln(\exp x) = \ln e^x$ shows that $\exp x = e^x$

7. ($\ln x \to \infty$)
   (a) Show geometrically that $\frac{1}{2} < \ln 2 < 1$
   (b) Prove that $\ln x$ is increasing
   (c) Using the log laws you proved above along with (a) and (b), show that if $x > 2^n$, then $\ln x > \frac{n}{2}$.
   (d) Explain why this shows that $\lim_{x\to\infty} \ln x = \infty$

Area Between Curves

1. Find the area between each of the following pairs of curves
   (a) $x + y^2 = 2; x + y = 0$
   (b) $y = |x|; y = x^2 - 2$
   (c) $x = y^2 - 4y; x = 2y - y^2$

2. Use a Riemann Sum (NOT antiderivatives) to find the area enclosed by the curves $y = x$ and $y = x^2$

3. Find the number a such that the line $x = a$ divides the region under the curve $y = \frac{1}{x^2}$ from $x = 1$ to $x = 4$ into two pieces of equal area.

4. (a) What is the area under the curve $y = \frac{1}{\sqrt{x}}$ between $x = 1$ and $x = b$, where b is some constant?

   (b) What happens to your answer to (a) as $b \to \infty$

   (c) Now repeat (a) and (b) with $y = \frac{1}{x^2}$