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Math 1A PDP Worksheet

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You should work on the following problems in groups of 3. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Max/Min Values

1. Suppose we have a function whose domain is each point on the surface of the earth, and the value at each point is the altitude of that point.
   (a) Where are the absolute maximum and minimum values attained?
   (b) What do local max/mins represent?
2. Find the critical numbers of each of the following functions:
   (a) $4x^3 - 9x^2 - 12x + 3$
   (b) $|2x + 3|$
   (c) $x^{4/5}(x - 4)^2$
3. Find the extreme values (and the places they are achieved) of each function for the given intervals
   (a) $e^{-x} - e^{-2x}$; $[0, 1]$
   (b) $3x^2 - 12x + 5$; $[0, 3]$
   (c) $x - \ln x$; $[\frac{1}{2}, 2]$
   (d) $\frac{x}{x^2+4}$; $[0, 3]$
4. Find the absolute maximum and minimum values of $f(x) = \sqrt{9 - x^2}$
5. A plane flying a constant speed of $500mi/h$ at an altitude of 1 mile flies directly over a radar station at time $t = 0$ and disappears over the horizon 30 minutes later.
   (a) At what time during the interval while the plane is visible to the radar tower is the distance between the tower and the plane changing the fastest? Slowest?
   (b) At what time is the angle between the ground, the radar station, and the plane changing the fastest? slowest?
6. During takeoff, the velocity of a certain rocket obeys the equation $v(t) = .0013t^3 - .09t^2 + 23t - 3$. If it takes 120 seconds to reach orbit, what is the maximum **acceleration** experienced by the rocket during takeoff?
7. True/False. For those that are true, explain why. For those that are false, provide a counterexample:
   (a) Every function has at least one local maximum or local minimum
   (b) Every function has an absolute maximum.
   (c) Every function has an absolute minimum on every closed interval
   (d) Every continuous function has an absolute maximum on every closed interval.
   (e) If c is a local extremum, then c is a critical point
   (f) If c is a critical point, then c is a local extremum
8. Prove that the function $x^{207} + x^{63} + x^3 + 9x - 10$ has no local extrema

Mean Value Theorem

1. Consider the function $f(x) = 1 - |x|$. Show that $f(-1) = f(1)$, but that there is no c such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

2. (a) If a function has "at most two roots," could it have just one root? How about no roots at all? Could it have three roots?
   (b) If a function has "at least two roots," could it have just one root? How about two roots? Three?
   (c) If a function has "at most two roots" and "at least two roots," how many roots does it have?

3. Show that the equation $x^5 + 3x^3 + 10x = 1$ has exactly one real solution.

4. Show that the equation $x^5 - 6x + c = 0$ (where c is a constant) has at most one real root in the interval $[-1, 1]$

5. Suppose $f(1) = 3$ and $2 \leq f'(x) \leq 6$ for $1 < x < 5$. Assuming f is continuously differentiable on $[1, 5]$, what are the possible values for $f(5)$?

6. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$ Hint: it's not enough to show that they have the same derivative. What else do you need?

7. It turns out that if $x > 0$, $\sqrt{1+x} < 1 + \frac{1}{2}x$. Let's prove it:
   (a) Start by letting $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$. What are $f(0)$ and $f(1)$?
   (b) Prove that if $b > 0, f(b) \neq 0$
   (c) Use (a) and (b) to conclude the original inequality

8. Generalize the method from the above problem to prove the following very useful theorem: If f, g are continous on $[a, b]$, differentiable on (a, b), $f(a) = g(a)$ and $f'(x) < g'(x)$ for all x in (a, b), then $f(x) < g(x)$ for all x in (a, b)

9. (a) Use the fact that a linear function has at most one root to show that a quadratic polynomial has at most two roots.
   (b) Now show a cubic has at most three roots.

10. A number a is called a fixed point of f if $f(a) = a$. Show that if $f'(x) \neq 1$, then f has at most 1 fixed point.