Some $\varepsilon - \delta$ proofs

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Basic Strategy

1. Write down what you're going to prove:
   - $\forall\varepsilon > 0 \exists\delta > 0 \ : \ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
   - $\forall\varepsilon > 0 \exists N \ : \ x > N \Rightarrow |f(x) - L| < \varepsilon$
   - $\forall M \exists\delta > 0 \ : \ 0 < |x - a| < \delta \Rightarrow f(x) > M$
   - $\forall M \exists N \ : \ x > N \Rightarrow f(x) > M$
2. Find δ or N
   (a) Locate $|x - a|$ or x
   (b) Turn everything else into constants by assuming $|x - a| < 1$ (or $\frac{1}{2}$ if that doesn't work) or $x > 1$
   (c) Then $\delta = \min(1, \text{the } \delta \text{ you found})$ or $N = max(1, \text{the } N \text{ you found})$ (use min for $-\infty$)
3. Prove your δ or N works. This basically just means doing the same things over again but in the opposite order, and justifying each replacement based on your choice of δ or N

Examples

In addition to the examples in the book and quiz solutions that are online[1], here's a few for reference:

1. $\lim\limits_{x \to 2} x^2 + x = 6$

   We will show that $\forall\varepsilon > 0 \exists\delta > 0 : 0 < |x - 2| < \delta \Rightarrow |x^2 + x - 6| < \varepsilon$

   Note $|x^2 + x - 6| = |x - 2| \cdot |x + 3|$, so we just need to turn $|x + 3|$ into a constant.

   If $|x - 2| < 1$, then $1 < x < 3$, so $4 < x + 3 < 6$ and thus $|x + 3| < 6$

   So $|x^2 + x - 6| < 6|x - 2|$ so we'll guess $\delta = \min(1, \varepsilon/6)$

   Claim: $\forall\varepsilon > 0, \ 0 < |x - 2| < \delta = \min(1, \varepsilon/6) \Rightarrow |x^2 + x - 6| < \varepsilon$

   Proof:

   If $|x - 2| < \delta$, then $|x - 2| < 1$, so we know by previous work that $|x + 3| < 6$. Since $|x - 2| < \delta$ we also know $|x - 2| < \varepsilon/6$. Then we have:

   $$|x^2 + x - 6| = |x - 2||x + 3| < 6|x - 2| < 6\frac{\varepsilon}{6} = \varepsilon$$

   as was to be shown.

2. $\lim\limits_{x \to \infty} \sqrt{x + 4} = \infty$

   We will show that $\forall M \exists N : x > N \Rightarrow \sqrt{x + 4} > M$

   Let M be given. Then we want $\sqrt{x + 4} > M$, which will happen if $x + 4 > M^2$, which will happen if $x > M^2 - 4$, so we guess $N = M^2 - 4$

   Claim: $\forall M, \ x > N^2 - 4 \Rightarrow \sqrt{x + 4} > M$

   Proof: Suppose $x > N = M^2 - 4$. Then $x + 4 > M^2$, so $\sqrt{x + 4} > |M| \geq M$, as was to be shown.

[1] See http://math.berkeley.edu/∼rbayer

3. $\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$

We will show that $\forall \varepsilon > 0 \exists N : x > N \Rightarrow \left|\frac{\sin x}{x} - 0\right| < \varepsilon$

Let's find N: Since $-1 < \sin x < 1$, $\left|\dfrac{\sin x}{x}\right| \leq \left|\dfrac{1}{x}\right|$. In order to drop the abs value bars, let's assume $x > 1$ To get $\left|\frac{1}{x}\right| < \varepsilon$, we need $x > \frac{1}{\varepsilon}$, so we'll guess $N = \max(1, \frac{1}{\varepsilon})$

Claim: $\forall \varepsilon > 0,\ x > \max(1, \frac{1}{\varepsilon}) \Rightarrow \left|\frac{\sin x}{x} - 0\right| < \varepsilon$

Proof: Suppose $x > \max(1, \frac{1}{\varepsilon})$. Then we have $x > 1$, so $|\frac{1}{x}| = \frac{1}{x}$ and

$$\left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{1/\varepsilon} = \varepsilon$$

as was to be shown. (Note that since $x > \frac{1}{\varepsilon}$, replacing the denominator x by $\frac{1}{\varepsilon}$ makes the whole thing bigger.

4. $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$

We'll show $\forall M \exists \delta :\ 0 < |x - 0| < \delta \Rightarrow \frac{1}{x^2} > M$

Since we're trying to show this limit goes to $+\infty$, we can safely assume $M > 0$. Then since we want $\frac{1}{x^2} > M$, we can cross multiply (note all terms are positive) to get $\frac{1}{M} > x^2$. Taking square roots gives $\sqrt{1/M} > |x|$, we we'll guess $\delta = \sqrt{1/M}$

Claim: $\forall M, 0 < |x| < \sqrt{1/M} \Rightarrow \frac{1}{x^2} > M$

If $|x| < \sqrt{1/M}$, then squaring both sides gives $x^2 < 1/M$, so cross multiplying (again: we may assume $M > 0$) gives $M < \frac{1}{x^2}$, as was to be shown.

5. $f(x) = \frac{1}{x}$ is continuous at 1

We'll show that $\lim_{x\to 1} f(x) = f(1) = 1$ by showing that $\forall \varepsilon > 0 \exists \delta > 0 :\ 0 < |x - 1| < \delta \Rightarrow |\frac{1}{x} - 1| < \varepsilon$

$$\begin{aligned} \left|\frac{1}{x} - 1\right| &= \left|\frac{1-x}{x}\right| \\ &= \frac{|x-1|}{|x|} \end{aligned}$$

If $|x - 1| < 1$, we get $0 < x < 2$, so $0 < |x| < 2$, and $\frac{1}{2} < \frac{1}{|x|} < \infty$, which is not a helpful inequality. So let's try $|x - 1| < \frac{1}{2}$ instead:

Then $\frac{1}{2} < x < \frac{3}{2}$, so $\frac{2}{3} < \frac{1}{|x|} < 2$.

Thus $\frac{|x-1|}{|x|} < 2|x - 1|$ and we'll try $\delta = \min(\frac{1}{2}, \varepsilon/2)$

Claim: $\forall \varepsilon > 0, 0 < |x - 1| < \min(\frac{1}{2}, \varepsilon/2) \Rightarrow |\frac{1}{x} - 1| < \varepsilon$

Proof: If $|x - 1| < \min(\frac{1}{2}, \varepsilon/2)$, then we know $|x - 1| < \frac{1}{2}$, so by previous work, $\frac{1}{|x|} < 2$. We also know that $|x - 1| < \varepsilon/2$, so we can do the following:

$$\begin{aligned} \left|\frac{1}{x} - 1\right| &= \frac{|x-1|}{|x|} \\ &< 2|x-1| \\ &< 2\varepsilon/2 \\ &= \varepsilon \end{aligned}$$

as was to be shown. Thus, $\lim_{x\to 1} \dfrac{1}{x} = \dfrac{1}{1} = 1$, so $\frac{1}{x}$ is continuous at 1

6. $\lim_{x\to -1^+} \frac{5}{(x+1)^3} = \infty$

We'll show $\forall M \exists \delta:\ 0 < x - (-1) < \delta \Rightarrow \frac{5}{(x+1)^3} > M$

Note: the assumption that $0 < x - (-1)$ guarantees that $x > -1$.

Let's find δ:

We want $\frac{5}{(x+1)^3} > M$, which will happen when $\frac{5}{M} > (x+1)^3$. Note that since $x > -1$ and $M > 0$, multiplying and dividing by them did not change the direction of the inequality. Thus, we want $\sqrt[3]{5/M} > x + 1$, which means we want $x - (-1) < \sqrt[3]{5/M}$, so we choose $\delta = \sqrt[3]{5/M}$.

Claim: $\forall M > 0, 0 < x - (-1) < \sqrt[3]{5/M} \Rightarrow \frac{5}{(x+1)^3} > M$

Proof: Since $0 < x + 1 < \sqrt[3]{5/M}$, we can cube both sides to get $(x+1)^3 < \frac{5}{M}$. By our assumption that $0 < x + 1$ and $M > 0$ we can multiply and divide to get $M < \frac{5}{(x+1)^3}$ as was to be shown.