Ricci flows in higher dimensions

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Bibliography

- R. Bamler, *Entropy and heat kernel bounds on a Ricci flow background,* arXiv:2008.07093
- R. Bamler, *Compactness theory of the space of Super Ricci flows,* arXiv:2008.09298
- R. Bamler, *Structure theory of non-collapsed limits of Ricci flows,* arXiv:2009.03243

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Online class on Ricci flow this fall semester

14:10–15:30 (Pacific time)
August 27 – December 3

email me (`rbamler@berkeley.edu`) or check my webpage (`https://math.berkeley.edu/~rbamler/rfclass.html`) for Zoom ID

Motivation & History

Ricci flow $(M, (g_t)_{t\in[0,T)})$ on a compact manifold M^n:

$$\partial_t g_t = -2\,\mathrm{Ric}_{g_t}$$

Important Question: Understand the singularity formation if $T < \infty$ (and the long-time asymptotics if $T = \infty$)

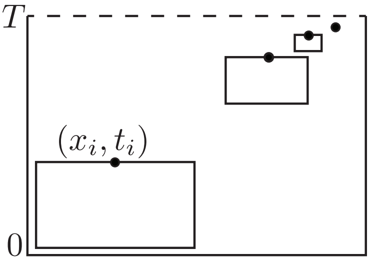
Blow-up analysis:

Choose $(x_i, t_i) \in M \times [0, T)$ s.t.:

$$t_i \nearrow T \qquad |\mathrm{Rm}|(x_i, t_i) \longrightarrow \infty$$

Hope that for some $\lambda_i \to \infty$:

$$\underbrace{(M, (\lambda_i^2 g_{\lambda_i^{-2}t+t_i}), x_i)}_{\text{parabolic rescaling}} \xrightarrow[i\to\infty]{} \underbrace{(M_\infty, (g_{\infty,t})_{t\le 0}, x_\infty)}_{\text{“singularity model”}}$$



So far: curvature bounds are necessary!

Dimension 2: singularity model $= (S^2, (2|t| g_{S^2})_{t<0})$ *(Chow, Hamilton)*

Dimension 3: singularity models are κ-solutions ... *(Perelman)*

Gradient shrinking soliton (M, g, f)**:** $\mathrm{Ric} + \nabla^2 f - \frac{1}{2} g = 0$
$\rightsquigarrow$ $g_t := |t| \phi_t^* g$ is RF, where $\phi_t =$ flow of $|t| \nabla f$, $t < 0$

- $|\mathrm{Rm}| \sim C/|t|$ (Type I)
- The singularity model of $(M, (g_t)_{t<0})$ is the flow itself.

Type-I curvature bound ($|\mathbf{Rm}| \leq \mathbf{C}/(\mathbf{T} - \mathbf{t})$): All singularity models are gradient shrinking solitons. *(Sesum, Naber, Enders, Buzano, Topping)*

Type-I scalar curvature bound ($\mathbf{R} \leq \mathbf{C}/(\mathbf{T} - \mathbf{t})$): All singularity models are gradient shrinking solitons with codimension ≥ 4 singular set.
(B., Chen, Hallgren, Wang, Zhang)

Folklore Conjecture

For a general Ricci flow "most" singularity models are gradient shrinking solitons.

Goal of this talk: Verify this conjecture in a certain (possibly optimal) sense.

Examples in higher dimensions

Appleton: $\exists$ RFs in dimension 4 whose blow-up limts are:

Eguchi-Hanson, $\mathbb{R}^4/\mathbb{Z}_2$, $\big($Bryant soliton$/\mathbb{Z}_2$, $\mathbb{R}P^3 \times \mathbb{R}\,\big)$

Ricci flat *singular*

gradient shrinking soliton

Stolarski: $\exists$ RFs in dimensions $n \geq 13$ whose only gradient shrinking soliton blow-up limit is a Ricci flat cone

Li, Tian, Zhu: $\exists$ Kähler-RF that has to develop a singularity, but cannot converge to a smooth gradient shrinking soliton.

Conclusion: Need to allow singular set in Folklore Conjecture + Ricci flat cones

Recall: Einstein metrics

Consider a sequence of pointed, complete Einstein manifolds (M_i^n, g_i, x_i), $\mathrm{Ric} = \lambda_i g_i$, $|\lambda_i| \leq 1$. Then a subsequence Gromov-Hausdorff converges to a pointed metric length space:

$$(M_i^n, g_i, x_i) \xrightarrow[i\to\infty]{GH} (X, d, x_\infty).$$

Suppose that the following non-collapsing condition holds:

$$|B(x_i, r)| \geq v > 0.$$

Then there is a regular-singular decomposition

$$X = \mathcal{R} \,\dot\cup\, \mathcal{S}$$

such that:

- $\mathcal{R}$ is an open manifold and there is a smooth Einstein metric g_∞ on $\mathcal{R}$ such that $d|_{\mathcal{R}} = d_{g_\infty}$. So (X, d) is isometric to the metric completion of $(\mathcal{R}, g_\infty)$.
- $\dim_{\mathcal{M}} \mathcal{S} \leq n-4$ *(Cheeger, Colding, Tian, Naber)*
- Any tangent cone at any point of X is a metric cone. *(Cheeger, Colding)*
- There is a stratification $\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-4} = \mathcal{S}$ such that $\dim_{\mathcal{H}} \mathcal{S}^k \leq k$ and every $x \in \mathcal{S}^k$ has a tangent cone that splits of an $\mathbb{R}^k$-factor. *(Cheeger, Naber)*

Main results of this talk

Similar theory for minimal surfaces, harmonic maps, mean curvature flow, harmonic map heat flow, . . .

Key points:

- There is a compactness and partial regularity theory for Ricci flow that is comparable to (and implies) that of Einstein metrics.
- This theory allows us to establish the Folklore Conjecture and several other related results.
- We need new, parabolic versions of notions such as: "metric space", "Gromov-Hausdorff limit", . . .

Theorem (B. 2020) Compactness theory of Ricci flows

Consider a sequence of n-dimensional, pointed Ricci flows:

$$(M_i, (g_{i,t})_{t \in (-T_i, 0]}, (x_i, 0)), \qquad T_\infty := \lim_{i \to \infty} T_i > 0.$$

Then a subsequence $\mathbb{F}$-converges to a metric flow over $(-T_\infty, 0]$:

$$(M_i, (g_{i,t})_{t \in (-T_i, 0]}, (\nu_{x_i,0})) \xrightarrow[i \to \infty]{\mathbb{F}} (\mathcal{X}, d, (\nu_{x_\infty})).$$

Suppose that the following non-collapsing condition holds:

$$\mathcal{N}_{x_i,0}(\tau_0) \geq -Y_0 > -\infty.$$

Then we have a regular-singular decomposition

$$\mathcal{X} = \mathcal{R} \,\dot{\cup}\, \mathcal{S}$$

such that:

- $\mathcal{X}$ restricted to $\mathcal{R}$ is given by a smooth Ricci flow spacetime structure and $\mathcal{X}$ is uniquely determined by this structure.
- $\dim_{\mathcal{M}^*} \mathcal{S} \leq (n+2) - 4$
- All tangent flows of $\mathcal{X}$ are gradient shrinking solitons with singularities.
- There is a filtration $\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-2} = \mathcal{S}$ such that $\dim_{\mathcal{H}^*} \mathcal{S}^k \leq k$ and every $x \in \mathcal{S}^k$ has a tangent flow that splits off an $\mathbb{R}^k$-factor or is static and splits off an $\mathbb{R}^{k-2}$-factor.

Consequences & Further results

Regarding Folklore Conjecture:

Theorem (B. 2020)

Consider a Ricci flow $(M, (g_t)_{t\in[0,T)})$, $T < \infty$. Then there is a metric space

$$(M_T, d_T) \qquad \text{“}= \lim_{t \nearrow T} (M, g_t)\text{”}$$

such that:

- If $g_t \to g_T$ smoothly on $U \subset M$, then $U \subset M_T$ and $d_T|_U$ is locally isometric to $d_{g_t}|_U$.
- For any “$(x_i, t_i) \to (z, T)$”, $z \in M_T$, there is a sequence of blow-ups that converges to singular gradient shrinking soliton.
  This soliton can be viewed as the tangent flow at (z, T).

In dimension 4:

Theorem (B. 2020)

In dimension 4 all tangent flows are given by singular gradient shrinking solitons on smooth orbifolds with conical singularities, i.e. (M, g, f), $\mathrm{Ric} + \nabla^2 f - \frac{1}{2} g = 0$. Moreover, either $R > 0$ or $(M, g) \cong \mathbb{R}^4/\Gamma$.

Regarding long-time asymptotics:

Theorem (B. 2020)

If $(M,(g_t)_{t\geq 0})$ is immortal, then for $Y, t \gg 1$

$$M = M_{\text{thick}}(t) \quad \dot\cup \quad M_{\text{thin}}(t)$$

such that:

- If $x_i \in M_{\text{thick}}(t_i)$ and $t_i \to \infty$, then $(M,(t_i^{-1} g_{t_i\, t}), x_i)$ converges to a singular, Einstein Ricci flow with $\text{Ric} = -\frac{1}{2t} g_{\infty,t}$.
  If $n = 4$, then this flow is given by an Einstein orbifold.
- If $x \in M_{\text{thin}}(t)$, then $\mathcal{N}_{x,t}(t/2) \leq -Y$.

Application: Backwards Pseudolocality

Theorem (B. 2020)

If $[t_0 - r^2, t_0] \subset I$ and
$$|B(x_0, t_0, r)| \geq \alpha r^n$$
and
$$|\mathrm{Rm}| \leq (\alpha r)^{-2} \quad \text{on} \quad B(x_0, t_0, r),$$
then $|\mathrm{Rm}| \leq (\varepsilon(n, \alpha) r)^{-2}$ on $P(x_0, t_0; \varepsilon r, -(\varepsilon r)^2)$.

Further Remarks:

- In dimension 3, this theory essentially recovers Perelman's theory.
- Compactness theory (not assuming non-collapsing) also holds for super Ricci flows $\partial_t g_t + 2\,\mathrm{Ric} \geq 0$.

Heat kernels on Ricci flow backgrounds

Let $(M, (g_t)_{t \in I})$ be a Ricci flow and $u, v \in C^2(M \times I)$.

Heat equation: $\square u = (\partial_t - \triangle_{g_t}) u = 0$

Conjugate heat equation: $\square^* v = (-\partial_t - \triangle_{g_t} + R_{g_t}) v = 0$

Heat kernel: $K(x, t; y, s)$, $x, y \in M$, $s < t$

for fixed (y, s): $\square K(\cdot, \cdot; y, s) = 0$, $\lim_{t \searrow s} K(\cdot, t; y, s) = \delta_y$

for fixed (x, t): $\square^* K(x, t; \cdot, \cdot) = 0$, $\lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x$

Representation formulas: If $\square u = \square^* v = 0$, then

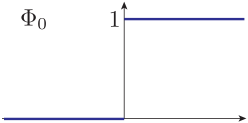
$$u(x, t) = \int_M K(x, t; \cdot, s) u(\cdot, s) dg_s \qquad v(y, s) = \int_M K(\cdot, t; y, s) v(\cdot, t) dg_t$$

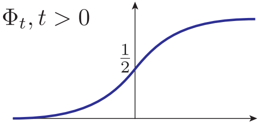
Reproduction formula for heat kernel: $s < t' < t$

$$K(x, t; y, s) = \int_M K(x, t; \cdot, t') K(\cdot, t'; y, s) dg_{t'}$$

Properties of heat equation:

- $u \leq C$ and $u \geq -C$ are preserved.
- $|\nabla u| \leq C$ is preserved
- Let $\Phi : \mathbb{R} \times \mathbb{R}_{\geq 0}$ be the solution the the 1-dimensional heat equation $\partial_t \Phi_t = \Phi_t''$ with initial condition $\Phi_0 = \chi_{[0,\infty)}$.





Improved gradient estimate (B. 2020)

If $0 < u(\cdot, t_0) < 1$, then for $t > t_0$

$$u_t(x) = \Phi_t(x') \qquad \Longrightarrow \qquad |\nabla u_t|(x) \leq \Phi'_{t-t_0}(x') \qquad (*)$$

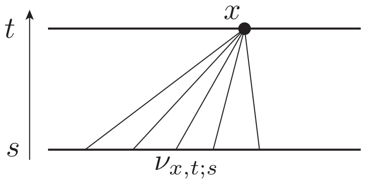
Moreover, $(*)$ is preserved for any fixed t_0.

Properties of conjugate heat equation:

- $v \geq 0$ is preserved
- $\int_M v(\cdot, s) dg_s$ is constant in s and $\int_M K(x,t;\cdot,s) dg_s = 1$
- Think of v as $d\mu_s = v(\cdot, s) dg_s$.

Conjugate heat kernel probability measure:

$$d\nu_{x,t;s} := K(x,t;\cdot,s)dg_s, \qquad \nu_{x,t;t} := \delta_x$$



Integral characterization of (conjugate) heat flows:

Heat flow: $\qquad \square u = 0 \qquad \Longleftrightarrow \qquad u(x,t) = \int_M u(\cdot,s)d\nu_{x,t;s}$

Conjugate heat flow:

$d\mu_s = v(\cdot,s)dg_s, \quad \square^* v = 0 \qquad \Longleftrightarrow \qquad \mu_s = \int_M \nu_{\cdot,t;s} d\mu_t$

Reproduction formula:

$$\nu_{x,t;s} = \int_M \nu_{\cdot,t';s} d\nu_{x,t;t'}$$

Metric flows

Metric flow over an interval I

$$\mathcal{X} = \big(\mathcal{X}, \mathfrak{t}, (d_t)_{t \in I}, (\nu_{x;s})_{x \in \mathcal{X}, s \in I, s \leq \mathfrak{t}(x)}\big)$$

1. $\mathcal{X}$ is a set consisting of **points**
2. $\mathfrak{t} : \mathcal{X} \to I$ is the **time-function** and its level sets $\mathcal{X}_t := \mathfrak{t}^{-1}(t)$ are **time-slices**
3. $(\mathcal{X}_t, d_t)$ is a complete and separable metric space for all $t \in I$
4. $\nu_{x;s}$ are probability measures called **conjugate heat kernel** and satisfy $\nu_{x;\mathfrak{t}(x)} = \delta_x$ and the **reproduction formula**

$$\nu_{x;s} = \int_{\mathcal{X}_t} \nu_{\cdot,t;s} d\nu_{x;t}$$

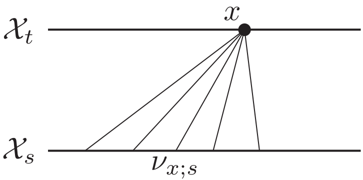
5. **(Conjugate) heat flows** are defined using the integral property as before.
6. We require that the improved gradient estimate holds for heat flows: If $u_{t_0} = \Phi_{t_0} \circ f_{t_0}$ for some 1-Lipschitz $f_{t_0} : \mathcal{X}_{t_0} \to \mathbb{R}$, then for all $t \geq t_0$ we have $u_t = \Phi_t \circ f_t$ for some 1-Lipschitz $f_t : \mathcal{X}_t \to \mathbb{R}$.

Ricci flow $(M, (g_t)_{t\in I})$ $\longrightarrow$ **Metric flow** $\mathcal{X}$

- $\mathcal{X} := M \times I$
- $\mathfrak{t} :=$ projection onto second factor.
- $d_t := d_{g_t}$ on $\mathcal{X}_t = M \times \{t\}$
- $d\nu_{(x,t);s} := K(x,t;\cdot,s)dg_s$

Note:

- The distance between points in different time-slices is not defined!
- This construction forgets worldlines $t \mapsto (x,t)$.
  Instead: For $x \in \mathcal{X}_t$ there is a probability distribution $\nu_{x;s}$ of points $y \in \mathcal{X}_s$ that lie in the "past" of x.



Concentration property

Variance of probability measure μ on a metric space (X, d):

$$\mathrm{Var}(\mu) := \int_X \int_X d^2(x,y)\, d\mu(x)\, d\mu(y)$$

Theorem (B. 2020)

On any Ricci flow

$$\mathrm{Var}(\nu_{x,t;s}) \leq H_n(t-s), \qquad (*)$$

where $H_n := \frac{(n-1)\pi^2}{2} + 4$.

A metric flow $\mathcal{X}$ is called H-**concentrated** if $(*) + \ldots$ holds for $H_n = H$.

"The past in $\mathcal{X}_s$ of any point $x \in \mathcal{X}_t$ is determined up to an error of $\sim \sqrt{t-s}$."

1-Wasserstein distance

μ_1, μ_2 probability measures on complete, separable metric space (X, d)

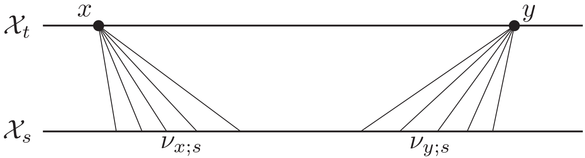
$$d_{W_1}(\mu_1, \mu_2) := \inf_{\substack{q \text{ coupling} \\ \text{btw } \mu_1, \mu_2}} \int_{X \times X} d \, dq = \sup_{\substack{f \,:\, X \to \mathbb{R} \\ \text{1-Lipschitz}}} \int_X f \, d(\mu_1 - \mu_2)$$

Lemma

If $x, y \in \mathcal{X}_t$, then for $s \leq t$ we have

$$d_{W_1}^{\mathcal{X}_s}(\nu_{x;s}, \nu_{y;s}) \leq d_t(x, y).$$

Moreover, $s \mapsto d_{W_1}^{\mathcal{X}_s}(\nu_{x;s}, \nu_{y;s})$ is non-decreasing and the same is true for any other pair of conjugate heat flows.

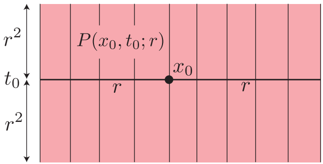


"Distances don't shrink on metric flows (in a probabilistic sense)"

Parabolic balls

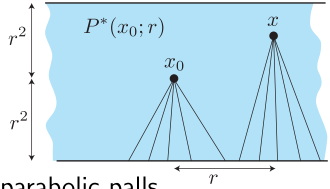
Conventional parabolic ball in a Ricci flow:

$$P(x_0,t_0;r) := B_{g_{t_0}}(x_0,r)\times[t_0-r^2,t_0+r^2]$$



P*-parabolic ball in a metric flow:

$$P^*(x_0;r) := \left\{ x\in\mathcal{X}_{t_0} : \begin{array}{c} \mathfrak{t}(x)\in[t_0-r^2,t_0+r^2] \\ d_{W_1}^{\mathcal{X}_{t_0-r^2}}(\nu_{x_0;t_0-r^2},\nu_{x;t_0-r^2})<r \end{array} \right\}$$



- standard containment properties still hold for P^*-parabolic palls (e.g. $P^*(x;r_1)\subset P^*(x;r_2)$ if $r_1\le r_2$)
- Conventional and P^*-parabolic balls are comparable if curvature bounded.
- The **natural topology** on $\mathcal{X}$ is generated by the set of all P^*-parabolic balls.
- P^*-parabolic balls allow the definition of the parabolic **Hausdorff and Minkowski dimension** $\dim_{\mathcal{H}^*}$ and $\dim_{\mathcal{M}^*}$.
  We count the time-direction twice!

Gromov-W_1-distance and convergence

Gromov-W_1-distance
If (X_i, d_i, μ_i), $i = 1, 2$, are two normalized metric measure spaces, then

$$d_{GW_1}\big((X_1, d_1, \mu_1), (X_2, d_2, \mu_2)\big) := \inf_{\varphi_1, \varphi_2, Z} d^Z_{W_1}((\varphi_1)_* \mu_1, (\varphi_2)_* \mu_2),$$

where the infimum is taken over all isometric embeddings $\varphi_i : (X_i, d_i) \to (Z, d_Z)$ into a common metric space (Z, d_Z).

Gromov-W_1-convergence

$$(X_i, d_i, \mu_i) \xrightarrow[i \to \infty]{GW_1} (X_\infty, d_\infty, \mu_\infty)$$

Important observation
Compare with pointed Gromov-Hausdorff convergence: The probability measures μ_i take the role of the basepoint.

$d_{\mathbb{F}}$-distance and $\mathbb{F}$-convergence

$d_{\mathbb{F}}$-distance:
Consider metric flows $\mathcal{X}_i$, $i = 1, 2$ equipped with conjugate heat flows $(\mu_{i,t})_{t \in I}$. We define

$$d_{\mathbb{F}}\big((\mathcal{X}^1, (\mu^1_t)_{t\in I}), (\mathcal{X}^2, (\mu^2_t)_{t\in I})\big)$$

to be the infimum over all $r > 0$ such that there are isometric embeddings

$$\big(\varphi^i_t : (\mathcal{X}^i_t, d^i_t) \to (Z_t, d^Z_t)\big)_{t \in I \setminus E, i=1,2}$$

with:

1. $|E| \le r^2$
2. $d^{Z_t}_{W_1}((\varphi^1_t)_* \mu^1_t, (\varphi^2_t)_* \mu^2_t) \le r$ for all $t \in I \setminus E$
3. "integral W_1-closeness of conjugate heat kernels between times $s, t \in I \setminus E$"

$\mathbb{F}$-convergence
If $d_{\mathbb{F}}\big((\mathcal{X}^i, (\mu^i_t)_{t\in I}), (\mathcal{X}^\infty, (\mu^\infty_t)_{t\in I})\big) \to 0$, then we write

$$(\mathcal{X}_i, (\mu_{i,t})_{t \in I_i}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X}_\infty, (\mu_{\infty,t})_{t\in I_i})$$

This implies Gromov-W_1-convergence at almost every time.

Let $\mathbb{F}_I$ be the space of pairs $(\mathcal{X}, (\mu_t)_{t \in I})$.

Theorem (B. 2020)

$(\mathbb{F}_I, d_{\mathbb{F}})$ is a complete metric space.

Suppose $I = (-T, 0]$. Fix n.

Theorem (B. 2020)

$$\left\{ \begin{array}{c} (\mathcal{X}, (\mu_t)_{t \in I}) \text{ corresponding to} \\ \text{Ricci flows } (M^n, (g_t)_{t \in I}, (\nu_{x,0;t})_{t \in I}) \end{array} \right\} \subset \mathbb{F}_I \quad \text{is precompact.} \qquad (*)$$

Corollary

For any sequence of n-dimensional, pointed Ricci flows $(M_i, (g_{i,t})_{t \in (-T,0]}, (x_i, 0))$ there is a subsequence such that:

$$(M_i, (g_{i,t})_{t \in (-T_i, 0]}, (\nu_{x_i,0})) \xrightarrow[i \to \infty]{\mathbb{F}} (\mathcal{X}, (\nu_{x_\infty})).$$

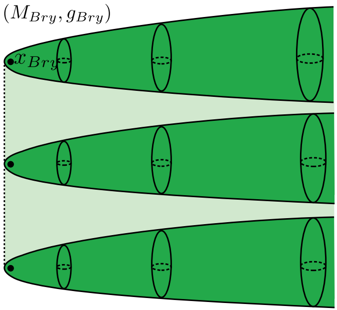
Remark: There is a compact subset $\mathbb{F}_I^*(H) \subset \mathbb{F}_I$, essentially corresponding to all H-concentrated metric flows, that contains the subset from $(*)$.

Digesting $\mathbb{F}$-convergence

If we assume curvature bounds, then: $\mathbb{F}$-convergence $\iff$ local smooth convergence in the sense of Cheeger, Gromov, Hamilton.

Example: Bryant soliton $(M_{\text{Bry}}, (g_{\text{Bry},t})_{t\in\mathbb{R}}, x_{\text{Bry}})$

- rotational symmetric
- $g_{\text{Bry},t} = dr^2 + f^2(r) g_{S^2}$,
  where $f(r) \sim \sqrt{r}$
- steady gradient soliton
  $\implies$ all time-slices are isometric



Consider blow-downs $(M_{\text{Bry}}, (\lambda_i^2 g_{\text{Bry},\lambda_i^{-2}t})_{t\in\mathbb{R}}, x_{\text{Bry}})$ for $\lambda_i \to 0$.

- Gromov-Hausdorff limit at any fixed time:
  $[0,\infty)$
- $\mathbb{F}$-limit:
  round shrinking cylinder $(S^2 \times \mathbb{R}, (g_t = 2|t| g_{S^2} + g_{\mathbb{R}})_{t<0})$
  this is the asymptotic soliton!

Ricci flow spacetimes

Ricci flow spacetime over an interval I:

$$\mathcal{M} = \big(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g\big)$$

1. $\mathcal{M}$ is a smooth $(n+1)$-manifold, called **spacetime manifold**
2. $\mathfrak{t} : \mathcal{M} \to I$ is a smooth map whose level sets $\mathcal{M}_t := \mathfrak{t}^{-1}(t)$ are called **time-slices**.
3. $\partial_{\mathfrak{t}}$ is a smooth vector field on $\mathcal{M}$ with $\partial_{\mathfrak{t}} \mathfrak{t} = 1.$ Its trajectories are **worldlines.**
4. g is a metric on the horizontal distribution $\ker d\mathfrak{t} \subset T\mathcal{M}$
5. **Ricci flow equation:** $\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2\,\mathrm{Ric}_g$

Ricci flow $(M, (g_t)_{t \in I})$ $\longrightarrow$ **Ricci flow spacetime** $\mathcal{M}$

- $\mathcal{M} := M \times I$
- $\mathfrak{t} :=$ projection onto second factor
- $\partial_{\mathfrak{t}} :=$ std. vector field on I
- $g := g_t$ on $\mathcal{M}_t = M \times \{t\}$

Structure of non-collapsed $\mathbb{F}$-limits

Let $\mathcal{X}$ be a $\mathbb{F}$-limit of smooth Ricci flows over I.
Assume the non-collapsing condition $\mathcal{N}_{x_i,0}(\tau_0) \geq -Y_0 > -\infty$.

Theorem (B. 2020)

There is a decomposition

$$\mathcal{X} = \mathcal{R} \,\dot\cup\, \mathcal{S}$$

and a smooth Ricci flow spacetime structure $(\mathcal{R}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ on $\mathcal{R}$ such that:

- $\mathcal{R} \subset \mathcal{X}$ is open and dense.
- For any $t \in I$ the time-slice $(\mathcal{X}_t, d_t)$ is the metric completion of $(\mathcal{R}_t, g_t)$.
- (Conjugate) heat flows restricted to $\mathcal{R}$ are uniquely characterized by $\square u = 0$ and $\square^* v = 0$ on $\mathcal{R}$.
- $\dim_{\mathcal{M}^*} \mathcal{S} \leq (n+2) - 4$
- Tangent flows at any $x \in \mathcal{X}$ ($= \mathbb{F}$-limits of blow-ups of $(\mathcal{X}, (\nu_{x,t}))$) are singular gradient shrinking solitons.
- There is a filtration $\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-2} = \mathcal{S}$ such that $\dim_{\mathcal{H}^*} \mathcal{S}^k \leq k$ and every $x \in \mathcal{S}^k$ has a tangent flow that splits off an $\mathbb{R}^k$-factor or is static and splits off an $\mathbb{R}^{k-2}$-factor.

Theorem (B. 2020)

If $\mathcal{X}$ is a gradient shrinking soliton, then there is an identification

$$\mathcal{X} = X \times I$$

for a metric space (X, d) with regular part $\mathcal{R}_X \subset X$ such that:

- $(\mathcal{X}_t, d_t) = (X, |t|^{1/2} d)$
- $(\mathcal{R}_t, g_t) = (\mathcal{R}_X, |t| g_{\mathcal{R}_X})$
- The soliton equation holds on $\mathcal{R}_X$.

If $n = 4$, then (X, d) is the length space of a smooth orbifold.

Outstanding promise: Non-collapsing condition

Pointed Nash entropy: *(Perelman, Topping, Hein, Naber)*

Fix $(x_0, t_0) \in M \times I$ and write $\tau := t_0 - t$, $\quad K(x_0, t_0; \cdot, \cdot) =: (4\pi\tau)^{-n/2} e^{-f}$

$$\mathcal{N}_{x_0,t_0}(\tau) := \int_M f(\cdot, t_0 - \tau) d\nu_{x_0,t_0;t_0-\tau} - \frac{n}{2}$$

Basic properties:

- $\mathcal{N}_{x_0,t_0}(\tau) \leq 0$
- $\frac{d}{d\tau}\mathcal{N}_{x_0,t_0}(\tau) \leq 0$
- There is a relation between $\mathcal{N}$ and Perelman's μ-entropy that implies: If $I = [0, T)$, then

  $$\mathcal{N}_{x_0,t_0}(\tau) \geq \mu[M, g_0, T] > -\infty.$$

  So a non-collapsing condition always holds on a fixed flow with $T < \infty$.

Guiding principle: On a manifold with $\mathrm{Ric} \geq -g$: $\dfrac{|B(x,r)|}{r^n} \approx e^{\mathcal{N}_x(r^2)}$

Theorem (B. 2020)

Suppose that $R \geq R_{\min}$. Set $\mathcal{N}_s^*(x,t) := \mathcal{N}_{x,t}(t-s)$.

1. $|\nabla \mathcal{N}_s^*| \leq \sqrt{\dfrac{n}{2(t-s)} - R_{\min}}$
2. $-\dfrac{n}{2(t-s)} \leq \square \mathcal{N}_s^* \leq 0$
3. (1)+(2) imply a bound on $\operatorname{osc} \mathcal{N}_s^*$ over P^*-parabolic neighborhoods.
4. For any (x,t), $s < t$, there is a point z near the "center" of $\nu_{x,t;s}$ such that
$$K(x,t;y,s) \leq \frac{C(\varepsilon)}{(t-s)^{n/2}} \exp\left(-\frac{d_s^2(y,z)}{(8+\varepsilon)(t-s)} \right)$$
5. $|B(x,t,r)| \leq C(R_{\min}) \exp(\mathcal{N}_{x,t}(r^2))$
6. Reverse lower volume bound holds near concentration centers of conjugate heat kernels and under scalar curvature bounds.
7. ...

The picture at the first singular time

Suppose that $(M, (g_t)_{t\in[0,T)})$ develops a singularity at time $T < \infty$.

Singular time-slice (M_T, d_T)**:**

$$M_T := \{\text{conjugate heat flows}(\mu_t)_{t\in[0,T)} \quad : \quad \text{Var}(\mu_t) \leq H_n(T-t)\}$$

$$d_T((\mu_t^1), (\mu_t^2)) := \lim_{t\nearrow T} d_{W_1}^{g_t}(\mu_t^1, \mu_t^2)$$

Theorem

- (M_T, d_T) is a complete metric space.
- If $g_t \to g_T$ on U as $t \nearrow T$, then $U \leftrightarrow U' \subset M_T$ and $d_{g_T} \cong d_T$ locally.
- For any $p := (\mu_t)$ any blow-ups of $(M, (g_t)_{t\in[0,T)}, (\mu_t)_{t\in[0,T)})$ subsequentially $\mathbb{F}$-converge to a singular gradient shrinking soliton.