Classification of diffeomorphism groups of 3-manifolds through Ricci flow

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Structure of Talk

- Part I: Diffeomorphism Groups
- Part II: Uniqueness of singular Ricci flows
- Part III: Applications of Ricci flow to diffeomorphism groups
- Part IV: Further Questions

Part I: Diffeomorphism Groups

Diffeomorphism groups — Basics

M mostly 3-dimensional compact manifold

Goal of this talk:

Understand $\quad \mathrm{Diff}(M) = \{\phi : M \to M \text{ diffeomorphism}\} \quad$ (C^∞-topology).

Main theme:

Pick a "nice" Riemannian metric g on M (e.g. constant sectional curvature) and compare $\mathrm{Diff}(M)$ with $\mathrm{Isom}(M)$.

$$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$$

Smale 1958

$O(3) = \mathrm{Isom}(S^2) \longrightarrow \mathrm{Diff}(S^2) \quad$ is a homotopy equivalence.

Proof: Later (using Ricci flow)

Diffeomorphism groups — Spherical case

Smale Conjecture

$O(4) = \mathrm{Isom}(S^3) \longrightarrow \mathrm{Diff}(S^3)$ is a homotopy equivalence.

Cerf 1964 (π_0), Hatcher 1983 (homotopy eq.)

Equivalent statements:

- The space of embeddings $S^2 \to \mathbb{R}^3$ is contractible.
- $\mathrm{Diff}(D^3 \text{ rel } \partial D^2)$ is contractible.
- The space of $K \equiv 1$ metrics on S^3 is contractible (more on this later).

Diffeomorphism groups — Spherical case

Generalized Smale Conjecture

$\mathrm{Isom}(S^3/\Gamma) \longrightarrow \mathrm{Diff}(S^3/\Gamma)$ is a homotopy equivalence.

S^3	Cerf 1964 / Hatcher 1983
$\mathbb{R}P^3$	unknown
lens spaces ($\neq \mathbb{R}P^3$)	Ivanov 1984, Hong, Kalliongis, McCullough, Rubinstein 2012
prism spaces (dihedral case)	
quaternionic spaces	
tetrahedral spaces	unknown (until now)
octahedral spaces	unknown (until now)
icosahedral spaces	unknown (until now)

Proofs are purely topological and technical.

Diffeomorphism groups — Non-spherical case

Hyperbolic case:

Gabai 2001

If M^3 is closed hyperbolic, then

$$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$$

is a homotopy equivalence.

General case:

M^3 is irreducible, geometric, non-spherical
$g =$ homogeneous metric of maximal symmetry.

Generalized Smale Conjecture for geometric manifolds

$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$ is a homotopy equivalence.

Solved, except for non-Haken infranil case
(Gabai, Ivanov, Hatcher, McCullough, Soma)

Diffeomorphism groups — Open questions

- Prove the remaining cases.
- Is there a unified, geometric/analytic proof of the known cases?

Main Results

Theorem A (Ba., Kleiner 2017)

The Generalized Smale Conjecture holds for all spherical space forms $M = S^3/\Gamma$ except for (possibly) $M = \mathbb{R}P^3$ (and S^3):

$$\mathrm{Diff}(M) \simeq \mathrm{Isom}(M) \qquad (*)$$

Theorem B (Ba., Kleiner 2017)

$(*)$ also holds for all closed hyperbolic 3-manifold M. (Gabai's Theorem)

Remarks:

- Proof via Ricci flow.
- Proof provides a uniform treatment of Thms A, B on fewer than 30 pages.
- Proof relies on Hatcher's Theorem for $M = S^3$.
- $M = \mathbb{R}P^3$ and $M = S^3$ (without Hatcher's Theorem) and other topologies still work in progress.
- Proof may be generalized to other geometric models.

$\mathrm{Diff}(M) \longleftrightarrow \mathrm{Met}(M)$

Space of Riemannian metrics: $\mathrm{Met}(M) = \{g \text{ metric on } M\}$

Space of const. curv. metrics: $\mathrm{Met}_{K\equiv k}(M) = \{g \in \mathrm{Met}(M) \mid K_g \equiv k\}$

Lemma

For any $g_0 \in \mathrm{Met}_{K\equiv k}(M)$:

$$\mathrm{Diff}(M) \simeq \mathrm{Isom}(M, g_0) \qquad \Longleftrightarrow \qquad \mathrm{Met}_{K\equiv k}(M) \text{ contractible}$$

Proof

$$\mathrm{Isom}(M, g_0) \longrightarrow \mathrm{Diff}(M) \longrightarrow \mathrm{Met}_{K\equiv k}(M)$$

$$\phi \longmapsto \phi^* g_0$$

is a fibration

Application: Diff(S^2) revisited

Ricci flow: $$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}, \qquad t \in [0, T)$$
$$g(0) = g_0$$

Hamilton, Chow: On S^2 we have $T < \infty$ and
$$(T - t)^{-1} g(t) \longrightarrow g_{\mathrm{round}} \qquad \text{with} \qquad K \equiv \tfrac{1}{2}$$

Smale 1958

$O(3) = \mathrm{Isom}(S^2) \longrightarrow \mathrm{Diff}(S^2)$ is a homotopy equivalence.

Proof (different from Smale's proof)

$$* \simeq \mathrm{Met}(S^2) \longrightarrow \mathrm{Met}_{K\equiv 1}(S^2)$$
$$g \longmapsto \text{limit of RF (modulo rescaling) with initial condition } g_0 = g$$

is a deformation retraction

$\Longrightarrow$ $\mathrm{Met}_{K\equiv 1}(S^2)$ is contractible

q.e.d.

Main result, reworded

Theorems A + B (Ba., Kleiner 2017)

If $M \not\approx \mathbb{R}P^3, S^3$, then $\mathrm{Met}_{K\equiv\pm1}(M)$ is contractible.

Idea of proof: Use Ricci flow to find a retraction $\mathrm{Met}(M) \longrightarrow \mathrm{Met}_{K\equiv\pm1}(M)$

Difficulty: Flow may not converge to a round metric at first singular time.
Surgeries?
Continuity of "round limits"?

Part II: Uniqueness of singular Ricci flows

Basics of Ricci flow

Ricci flow: $(M^n, g(t))$, $t \in [0, T)$

$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}, \qquad g(0) = g_0 \qquad (*)$$

Hamilton 1982

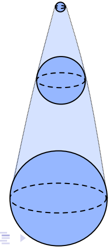
- $(*)$ has a unique solution $(g(t))_{t\in[0,T)}$ for maximal $T > 0$ if M is compact.
- If $T < \infty$, then

$$\lim_{t\to T} \max_M |\mathrm{Rm}_{g(t)}| = \infty$$

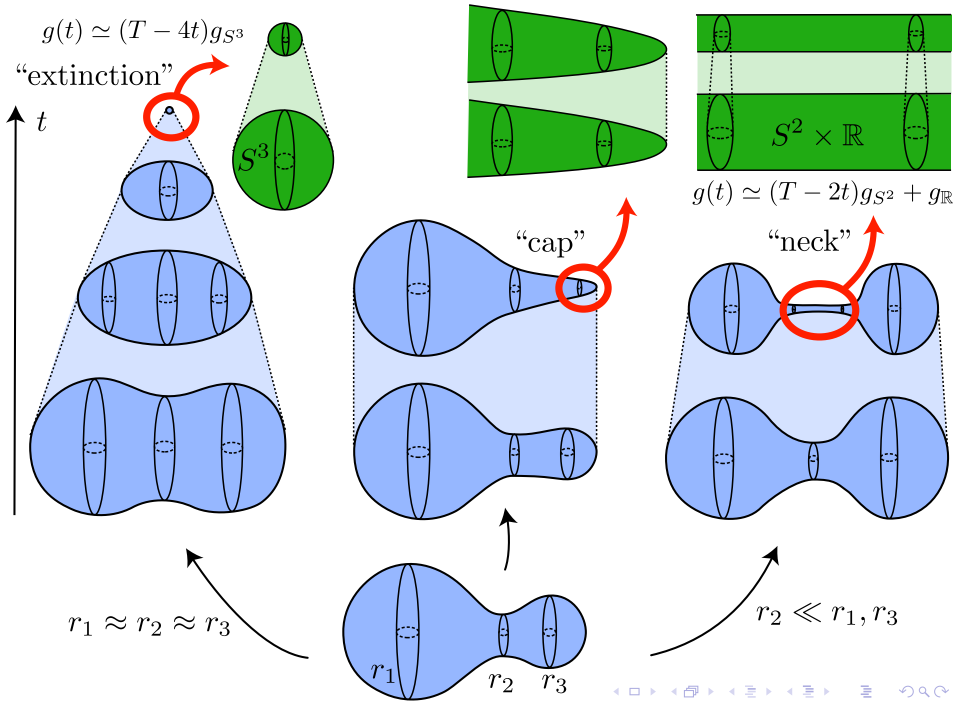
Speak: "$g(t)$ develops a singularity at time T".

Basic Example:

Round shrinking sphere $g(t) = 4(T-t)g_{\mathrm{round}}$ on $M = S^3$



Generally, flow develops non-round singularity!

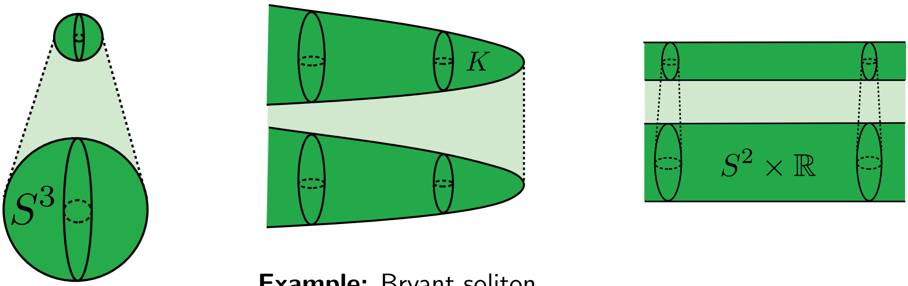

$g(t) \simeq (T - 4t)g_{S^3}$
"extinction"
t
S^3
"cap"
$S^2 \times \mathbb{R}$
$g(t) \simeq (T - 2t)g_{S^2} + g_{\mathbb{R}}$
"neck"
$r_1 \approx r_2 \approx r_3$
$r_2 \ll r_1, r_3$
r_1
r_2
r_3

Theorem (Perelman 2002), imprecise form

The singularity models in dimension 3 are κ-solutions.

Qualitative classification of κ-solutions

"extinction" "cap" "neck"



Example: Bryant soliton

Local Models & Canonical Neighborhood Assumption

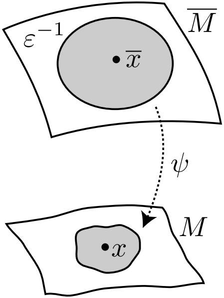
(M, g) **Riemannian manifold,** $x \in M$ **point**

- scale of $x = \rho(x) = |\mathrm{Rm}|^{-1/2}(x)$
- $(\overline{M}, \overline{g}, \overline{x})$ pointed model space
- $(\overline{M}, \overline{g}, \overline{x})$ local ε-model at x if there is

$$\psi : B(\overline{x}, 0, \varepsilon^{-1}) \longrightarrow M$$

such that $\psi(\overline{x}) = x$ and

$$\left\| \rho^{-2}(x) \psi^* g - \overline{g} \right\|_{C^{[\varepsilon^{-1}]}} < \varepsilon.$$



$(M, g(t))$ **Ricci flow**

- ... satisfies ε-canonical neighborhood assumption at scales $< r_0$ if all (x, t) with $\rho(x, t) < r_0$ are locally ε-modeled on the final time-slice $(\overline{M}, \overline{g}(0), \overline{x})$ of a pointed κ-**solution.**

Theorem (Perelman 2002), precise form

$(M^3, g(t))$ satisfies the ε-canonical neighborhood assumption at scales $< r(\varepsilon)$.

Ricci flow with surgery

Given (M, g_0) construct
Ricci flow with surgery:

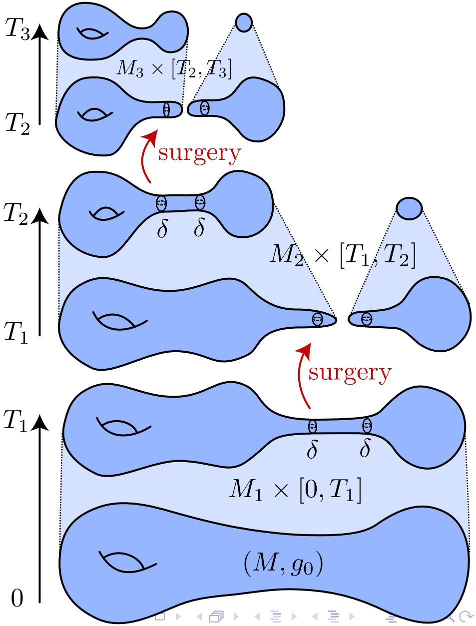
$$(M_1, g_1(t)), t \in [0, T_1],$$
$$(M_2, g_2(t)), t \in [T_1, T_2],$$
$$(M_3, g_3(t)), t \in [T_1, T_2], \quad \ldots$$

surgery scale $\approx \delta \ll 1$

Perelman 2003

- process can be continued indefinitely
- no accumulation of T_i.
- extinction if $\pi_1(M) < \infty$.

$M_k \approx$ connected sums components of M_{k+1} and copies of $S^2 \times S^1$ or S^3/Γ.



Ricci flow with surgery

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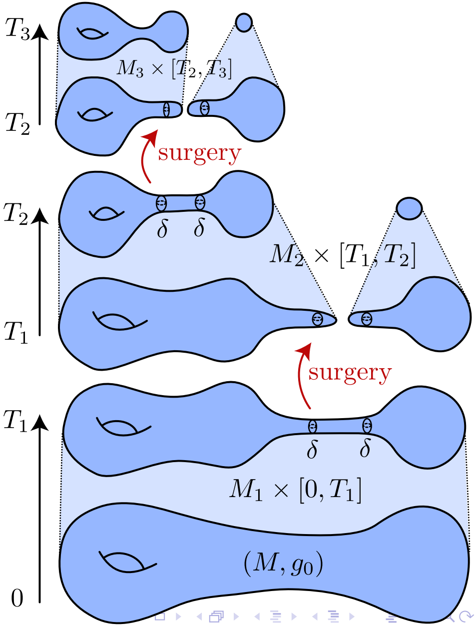
$$(M_3, g_3(t)), t \in [T_1, T_2], \quad \ldots$$

surgery scale $\approx \delta \ll 1$

high curvature regions are ε-close to κ-solutions:

- necks $\approx S^2 \times \mathbb{R}$
- spherical components
- caps

"ε-canonical neighborhood assumption"



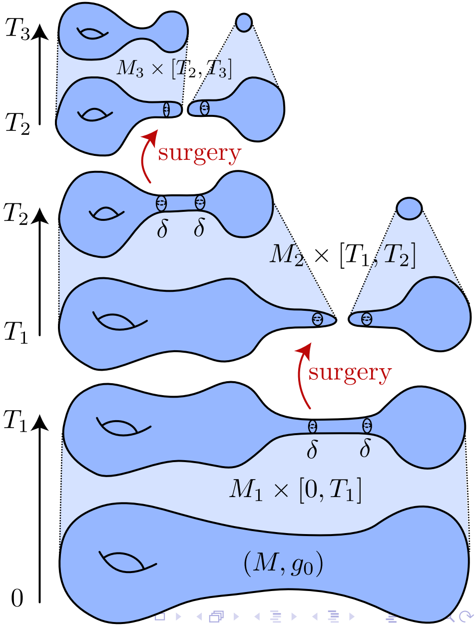
Ricci flow with surgery

Note:

surgery process is not canonical
(depends on surgery parameters)

Perelman:

- *It is likely that [. . .] one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that.*
- *Our approach [. . .] is aimed at eventually constructing a canonical Ricci flow, [. . .] - a goal, that has not been achieved yet in the present work.*

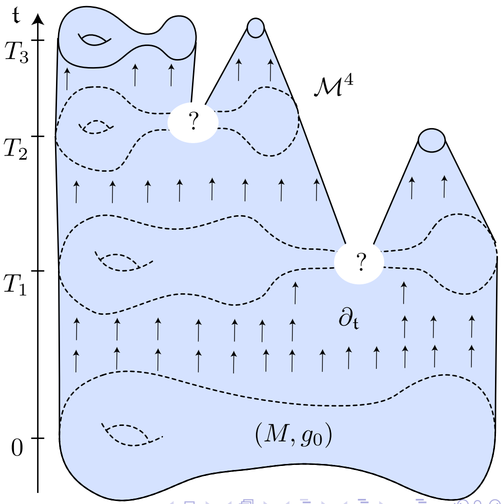


Space-time picture

- Space-time 4-manifold:

  $\mathcal{M}^4 = \big(M_1 \times [0, T_1] \;\cup\; M_2 \times [T_1, T_2] \;\cup\; M_3 \times [T_2, T_3] \;\cup\; \ldots\big) - \text{surgery points}$

- Time function: $\mathfrak{t} : \mathcal{M} \to [0, \infty)$.
- Time-slice: $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$
- Time vector field: $\partial_\mathfrak{t}$ on $\mathcal{M}$ (with $\partial_\mathfrak{t} \cdot \mathfrak{t} = 1$).
- Metric g: on the distribution $\{d\mathfrak{t} = 0\} \subset T\mathcal{M}$
- Ricci flow equation: $\mathcal{L}_{\partial_\mathfrak{t}} g = -2\,\mathrm{Ric}_g$



$(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ is called a Ricci flow space-time.

Note: there are "holes" at scale $\approx \delta$
space-time is δ-complete

Theorem (Kleiner, Lott 2014)

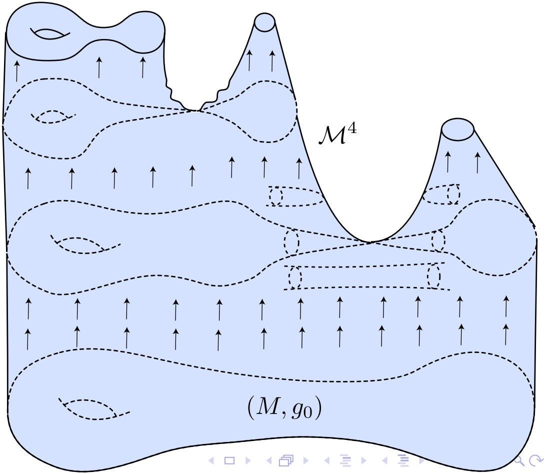
Given a compact (M^3, g_0), there is a Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ s.t.:

- initial time-slice: $(\mathcal{M}_0, g) = (M, g_0)$.
- $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is 0-complete (i.e. "singularity scale $\delta = 0$")
- $\mathcal{M}$ satisfies the ε-canonical nbhd assumption at small scales for all $\varepsilon > 0$.

$(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ flows "through singularities at infinitesimal scale"

Remarks:

- $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is smooth everywhere and not defined at singularities
- singular times may accummulate
- $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ arises as limit for $\delta_i \to 0$.



Uniqueness

Theorem (Ba., Kleiner, 2016)

There is a constant $\varepsilon_{\mathrm{can}} > 0$ such that:
Every Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is uniquely determined by its initial time-slice $(\mathcal{M}_0, g_0)$, provided that it

- is 0-complete and
- satisfies the $\varepsilon_{\mathrm{can}}$-canonical neighborhood assumption below some positive scale.

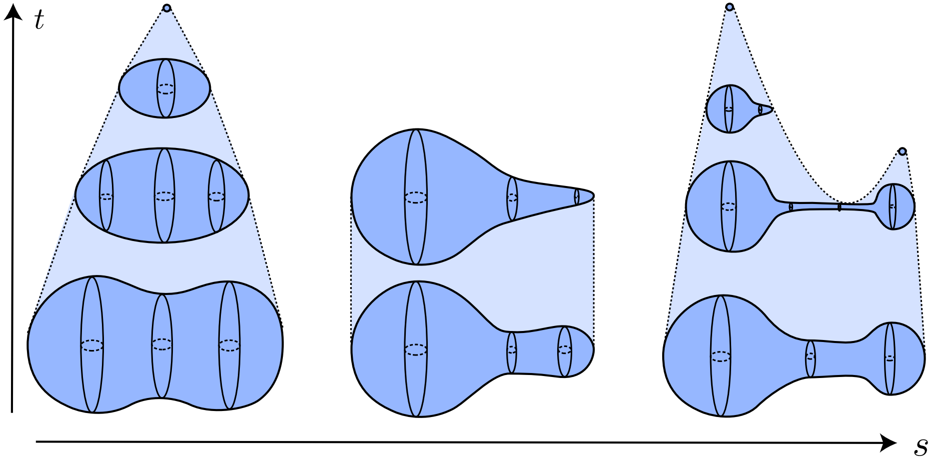
Corollary

For every compact (M^3, g_0) there is a unique, canonical singular Ricci flow space-time $\mathcal{M}$ with $\mathcal{M}_0 = (M^3, g_0)$.

Uniqueness $\longrightarrow$ Continuity

continuous family of metrics $(g^{(s)})_{s\in[0,1]}$ on M

$\rightsquigarrow \quad \{\mathcal{M}^{(s)}\}_{s\in[0,1]}$ singular RFs



Uniqueness $\longrightarrow$ Continuity

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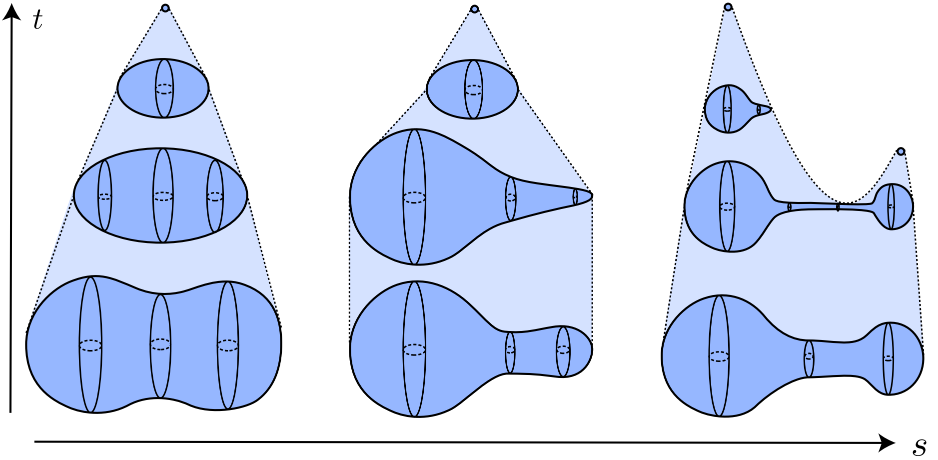

Uniqueness $\longrightarrow$ Continuity

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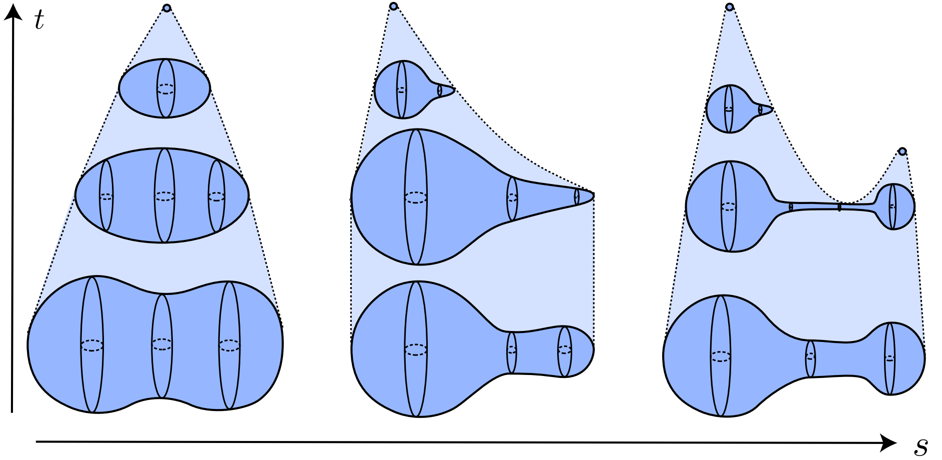


Continuity of singular RFs

Corollary

The singular Ricci flow space-time $\mathcal{M}$ depends continuously on its initial data $(\mathcal{M}_0, g_0)$ (in a certain sense).

Corollary

Every continuous/smooth family $(g^{(s)})_{s \in \Omega}$ of Riemannian metrics on a compact manifold M^3 can be evolved to a "continuous/smooth family of singular Ricci flows" $(\mathcal{M}^{(s)})_{s \in \Omega}$.

Part III: Applications to diffeomorphism groups

Setup

Assume $M = S^3/\Gamma$, $\Gamma \neq 1, \mathbb{Z}_2$ (hyperbolic case is similar)

Goal:

Theorem

$\mathrm{Met}_{K\equiv 1}(M)$ is contractible.

Strategy:

- **Hope:** Construct retraction

$$\mathrm{Met}(M) \longrightarrow \mathrm{Met}_{K\equiv 1}(M)$$

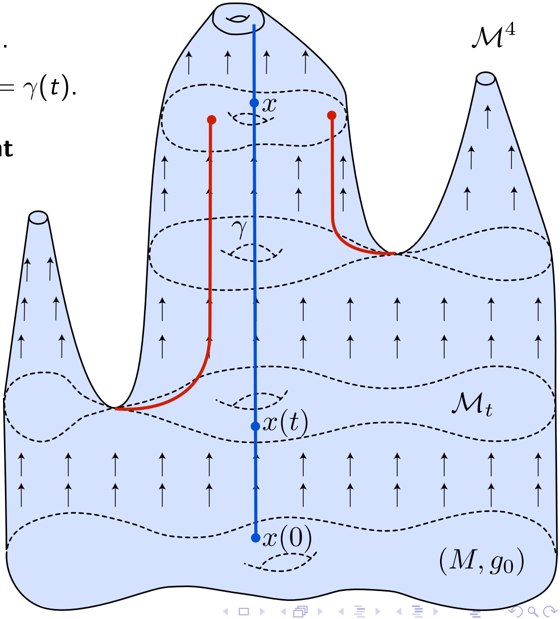
- For any $g \in \mathrm{Met}(M)$ there is a (unique) $\mathcal{M}^g$ with $(\mathcal{M}^g_0, g_0) = (M, g)$.
- $\mathcal{M}^g$ goes extinct in finite time and depends continuously on g.
- **Main step:** Construct $\widehat{g}^g \in \mathrm{Met}_{K\equiv 1}(M)$ based on $\mathcal{M}^g$.

Worldlines and bad points

Worldline: Trajectories γ of ∂_t.

If $\gamma(t_0) = x$, then we write $x(t) := \gamma(t)$.

A point $x \in M$ is called **bad point** if $x(0)$ does not exist (i.e. worldline is incomplete).



Lemma (Kleiner, Lott)

There are at most finitely many bad points in every component $\mathcal{C} \subset \mathcal{M}_t$ of every time-slice.

Fix $\mathcal{M} = \mathcal{M}^g$ with $(\mathcal{M}, g_0) = (M, g)$.

There are $T_g^1 < T_g^2$ such that:

- for every $t \in [T_g^1, T_g^2)$ there is a unique component $\mathcal{C}_t \subset \mathcal{M}_t$ with $\mathcal{C}_t \approx M$.
- $(\mathcal{C}_t, g_t)$ converges to a round metric as $t \nearrow T_g^2$ (modulo rescaling).

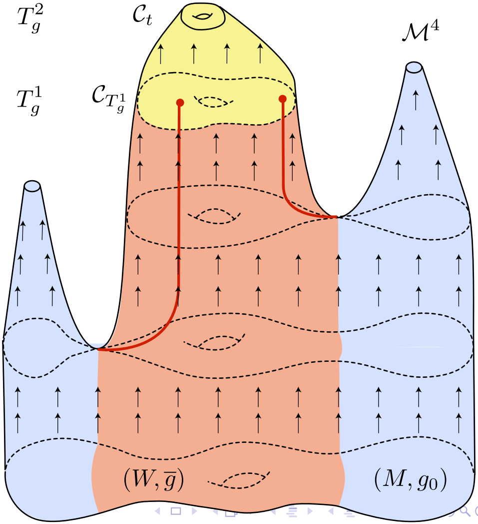
$W := \{x(0) \mid x \in \mathcal{C}_{T_g^1}\} \subset M$

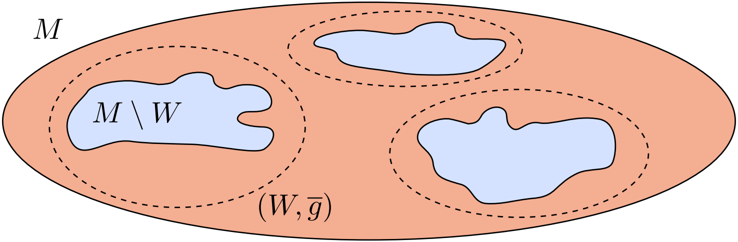
$\overline{g}_t :=$ pushforward of g_t onto W by flow of $-\partial_t$

$\overline{g}_t \xrightarrow[t \nearrow T_g^2]{} \overline{g}$ modulo rescaling ($K_{\overline{g}} \equiv 1$)

$(W, \overline{g}) \cong S^3/\Gamma \setminus \{p_1, \ldots, p_N\}$

($p_1, \ldots, p_N$ correspond to bad points)





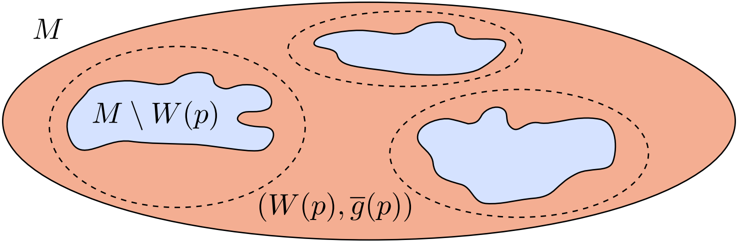
This process describes a continuous canonical map

$$\mathrm{Met}(M) \longrightarrow \mathrm{PartMet}_{K\equiv 1}(M)$$
$$g \longmapsto (W, \overline{g})$$

where $\mathrm{PartMet}_{K\equiv 1}(M)$ consists of pairs $(W, \overline{g})$ such that:

- $W \subset M$ open
- $(W, \overline{g})$ is isometric to the round punctured S^3/Γ
- $M \setminus W$ can be covered finitely many, pairwise disjoint disks
- If $K_g \equiv 1$, then $(W, \overline{g}) = (M, g)$.

Topology on $\mathrm{PartMet}_{K\equiv 1}(M)$: C^∞-convergence on compact subsets of W (not Hausdorff)



This process describes a continuous canonical map

$$\mathrm{Met}(M) \longrightarrow \mathrm{PartMet}_{K\equiv 1}(M)$$
$$g \longmapsto (W, \overline{g})$$

Proof that $\pi_k(\mathrm{Met}_{K\equiv 1}(M)) = 0$: Consider $g : S^k \longrightarrow \mathrm{Met}_{K\equiv 1}(M)$

1. extend g to continuous family $g' : D^{k+1} \longrightarrow \mathrm{Met}(M)$, $g'|_{\partial D^{k+1}} = g$
2. $\rightsquigarrow$ continuous family
$$(W(p), \overline{g}(p)) \in \mathrm{PartMet}_{K\equiv 1}(M), \qquad p \in D^{k+1}$$
such that $W(p) = M$ and $\overline{g}(p) = g(p)$ for $p \in \partial D^{k+1}$.
3. Replace $\overline{g}(p)$ with a $K \equiv 1$ metric on each disk (unique up to contractible ambiguity, due to Smale Conjecture for S^3). $\rightsquigarrow$ continuous family $\widehat{g}(p) \in \mathrm{Met}_{K\equiv 1}(M)$ by inductive construction over skeleta of D^{k+1}.

Part IV: Further Questions

Further Questions

- $\mathbb{R}P^3$ case
- Reprove Hatcher's Theorem (S^3 case)
- Generalized Smale Conjecture for other geometric manifolds.

PSC Conjecture

$\mathrm{Met}_{R>0}(S^3) = \{g \in \mathrm{Met}(S^3) \mid R_g > 0\}$ is contractible.

Marques 2012: $\pi_0(\mathcal{R}^+(S^3)) = 0$.

Necessary Tools:

- Better understanding of continuous families of singular Ricci flows
- Asymptotic characterization of the flow.
  In non-spherical case: Does the flow always converge to its geometric model?