Uniqueness of weak solutions to the Ricci flow

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(based on joint work with Bruce Kleiner, NYU)

September 2017

Structure of Lecture Series

1. Introduction
   - Preliminaries on Ricci flows
   - Statement of main results
   - Detailed analysis of Blow-ups
   - Introduction of toy case
2. Stability analysis
3. Comparing singular Ricci flows

slides available at `http://math.berkeley.edu/~rbamler/`

Preliminaries on Ricci flow

- Basics
- Singularity analysis (blow-ups, κ-solutions etc.)
- Ricci flow with surgery
- Kleiner and Lott's construction of RF spacetimes

Ricci flow, basics

Ricci flow: $\big(M^n, (g_t)_{t\in[0,T)}\big)$

$$\partial_t g_t = -2\,\mathrm{Ric}_{g_t}, \qquad g_0 = g \tag{$*$}$$

Hamilton 1982

- If M is compact, then $(*)$ has a unique solution.
- If $T < \infty$, then

$$\max_M |\mathrm{Rm}_{g_t}| \xrightarrow{t \nearrow T} \infty$$

Speak: "(g_t) *develops a singularity at time* T".

Goal of this talk

Theorem (Ba., Kleiner, 2016)

Any (compact) 3-dimensional (M^3, g) can be evolved into a **unique** (canonical), weak Ricci flow defined for all $t \geq 0$ that "flows through singularities".

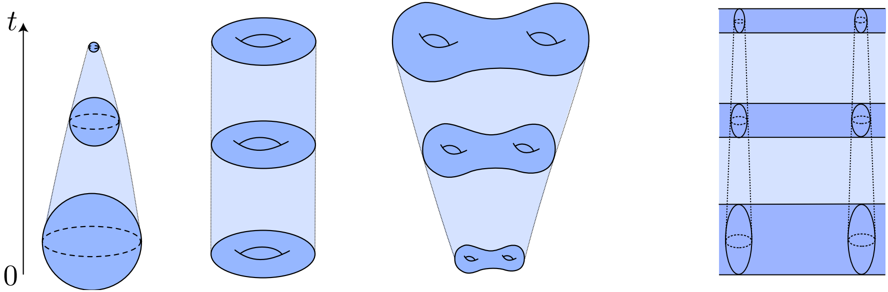
Ricci flow, basics

Ricci flow: $(M^n, (g_t)_{t\in[0,T)})$

$$\partial_t g_t = -2\,\mathrm{Ric}_{g_t}, \qquad g_0 = g \tag{$*$}$$

Important Examples: If $\mathrm{Ric}_g = \lambda g$, then $g_t = (1-2\lambda t)g$

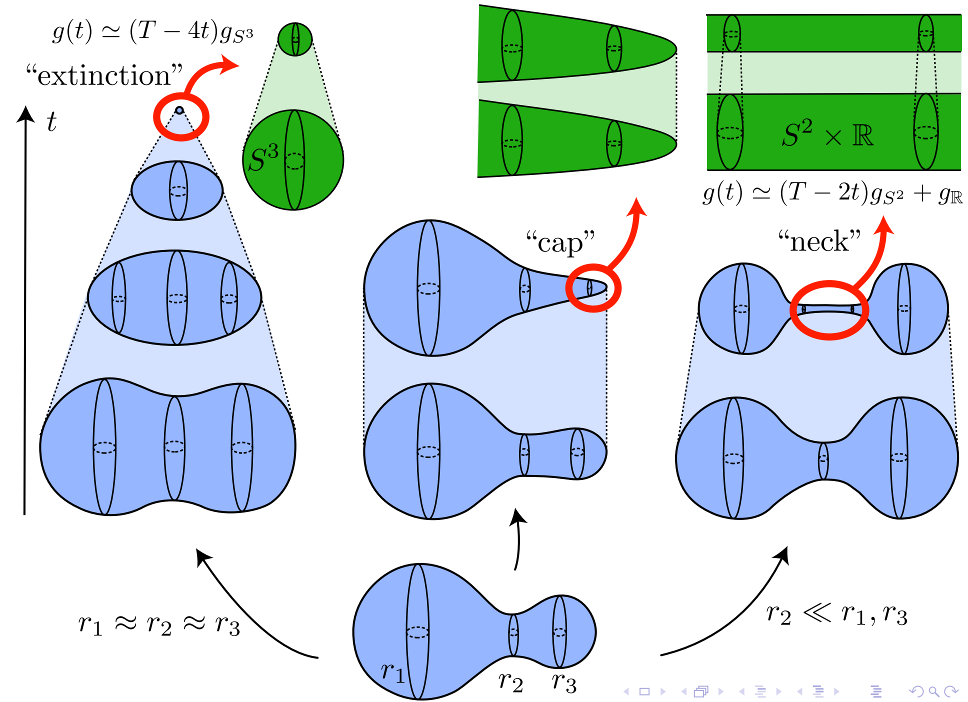
$\lambda > 0$ $\qquad$ $\lambda = 0$ $\qquad$ $\lambda < 0$ $\qquad$ round shrinking cylinder



$T = \frac{1}{2\lambda}$

$M = S^2 \times \mathbb{R}$

$g_t = -2tg_{S^2} + g_{\mathbb{R}}$



$g(t) \simeq (T - 4t) g_{S^3}$
"extinction"
t
S^3
$S^2 \times \mathbb{R}$
$g(t) \simeq (T - 2t) g_{S^2} + g_{\mathbb{R}}$
"cap"
"neck"
$r_1 \approx r_2 \approx r_3$
$r_2 \ll r_1, r_3$
r_1
r_2
r_3

Singularity analysis, general case

General case:

Perelman 2002, very imprecise form

"These are (essentially) all 3d singularities"

To make this statement more precise, we have to discuss:

- the No Local Collapsing Theorem
- geometric convergence of RFs (blow-up analysis)
- qualitative classification of κ-solutions

Singularity analysis, general case

Consider a general Ricci flow $(M^n, (g_t)_{t \in [0,T)})$

Goal: "Rule out $\approx S^1(\varepsilon) \times \Sigma^2$-singularity models"

No Local Collapsing Theorem (Perelman 2002)

There is a constant $\kappa(n, T, g_0) > 0$ such that $(M^n, (g_t)_{t \in [0,T)})$ is κ-noncollapsed at scales < 1, i.e.

for any $(x,t) \in M \times [0,T)$ and $0 < r < 1$:

$$|\mathrm{Rm}|(\cdot, t) < r^{-2} \quad \text{on} \quad B(x,t,r) \qquad \Longrightarrow \qquad |B(x,t,r)|_t \geq \kappa r^n$$

Corollary

$$|\mathrm{Rm}|(\cdot, t) < r^{-2} \quad \text{on} \quad B(x,t,r) \qquad \Longrightarrow \qquad inj(x,t) > c(\kappa) r^n$$

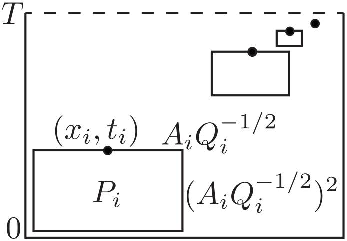
Singularity analysis, general case

Blowup analysis: Choose $(x_i, t_i) \in M \times [0, T)$ s.t.

$Q_i := |\mathrm{Rm}|(x_i, t_i) \xrightarrow[i\to\infty]{} \infty$

$$|\mathrm{Rm}| < CQ_i \quad \text{on} \quad P_i = P(x_i, t_i, A_i Q_i^{-1/2}) = B(x_i, t_i, A_i Q_i^{-1/2}) \times [t_i - (A_i Q_i^{-1/2})^2, t_i]$$

for $A_i \to \infty$ "parabolic ball"



parabolic rescaling: $g^i_t := Q_i g_{Q_i^{-1} t + t_i}$ $\quad (|\mathrm{Rm}| < C$ on $P(x_i, 0, A_i)$, NLC $\Longrightarrow \mathrm{inj}(x_i, 0) > c > 0)$

After passing to a subsequence:

$$(M, g^i_0, x_i) \xrightarrow[i\to\infty]{C^\infty-CG} (\overline{M}, \overline{g}_0, \overline{x})$$

i.e. $\exists$ diffeomorphisms onto their images $\psi_i : B^{\overline{M}}(\overline{x}, 0, A'_i) \to M_i$, $\psi_i(\overline{x}) = x_i$, $A'_i \to \infty$ s.t.

$$\psi_i^* g^i_0 \xrightarrow[i\to\infty]{C^\infty_{loc}} \overline{g}$$

Singularity analysis, general case

After passing to a subsequence:

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$$\psi_i^* g_0^i \xrightarrow[i\to\infty]{C^\infty_{loc}} \overline{g}$$

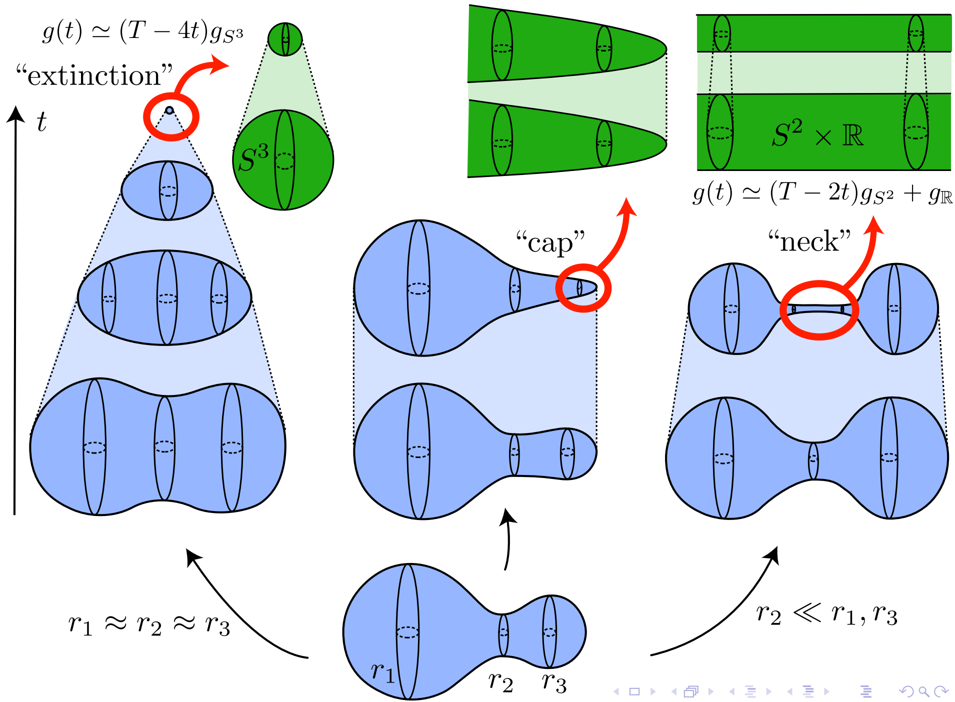
Using $|\mathrm{Rm}| < C$ on $P(x_i, 0, A_i)$, we can show that $|\partial^m \psi_i^* g_t^i|$ are locally uniformly bounded for all i. So after passing to a subsequence

$$\psi_i^* g_t^i \xrightarrow[i\to\infty]{C^\infty_{loc}} \overline{g}_t,$$

where $(\overline{g}_t)_{t\in(-\infty,0]}$ is an ancient Ricci flow. $\rightsquigarrow$ Hamilton's convergence of RFs:

$$(M, (g_t^i)_{t\in[-TQ_i,0]}, x_i) \xrightarrow[i\to\infty]{C^\infty - HCG} (\overline{M}, (\overline{g}_t)_{t\in(-\infty,0]}, \overline{x})$$

"$(\overline{M}, (\overline{g}_t)_{t\in(-\infty,0]}, \overline{x})$ models the flow near (x_i, t_i) for large i"



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r_1
r_2
r_3

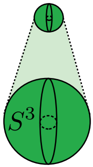
Singularity analysis, general case

Rotationally symmetric example

Extinction $\longrightarrow$ shrinking sphere

$\overline{M} = S^3$,

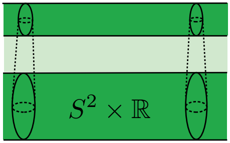
$\overline{g}_t = -4t g_{S^3}$



Neck pinch $\longrightarrow$ shrinking cylinder

$\overline{M} = S^2 \times \mathbb{R}$,

$\overline{g}_t = -2t g_{S^2} + g_{\mathbb{R}}$

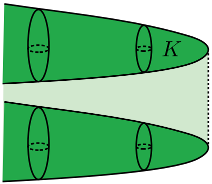


Cap $\longrightarrow$ Bryant soliton

$(\overline{M}, (\overline{g}_t)) \cong (M_{Bry}, (g_{Bry,t})_{t \in \mathbb{R}})$

$M_{Bry} = \mathbb{R}^3$, $\qquad g_{Bry,t} = dr^2 + f_t^2(r) g_{S^2}$

$f_t(r) \sim \sqrt{r} \quad$ as $\quad r \to \infty$



steady soliton equation: $\quad \mathrm{Ric} = \nabla^2 f = \frac{1}{2} \mathcal{L}_{\nabla f} g \quad \Longrightarrow \quad g_{Bry,t} = \Phi_t^* g_{Bry,0}$

Singularity analysis, general case

General case:

Perelman 2002, imprecise form

All blow-ups $(\overline{M}, (\overline{g}_t)_{t \in (-\infty, 0]})$ are κ-solutions.

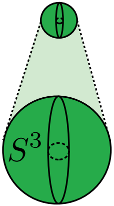
κ-solution: ancient flow $(M, (g_t)_{t \in (-\infty, 0]})$ s.t.

- $sec \geq 0, \quad R > 0, \quad |\mathrm{Rm}| < C \quad$ on $\quad M \times (-\infty, 0]$
- κ-noncollapsed at all scales:
  $|\mathrm{Rm}| < r^{-2}$ on $B(x,t,r) \quad \Longrightarrow \quad |B(x,t,r)|_t \geq \kappa r^3$

Singularity analysis, general case

Qualitative classification of κ-solutions

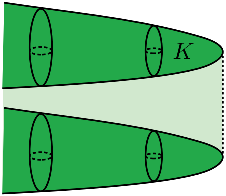
"extinction"



$M \approx S^3/\Gamma$

isometry types
not classified

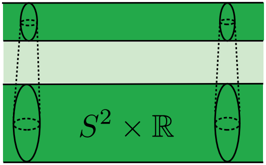
"cap"



$M \approx \mathbb{R}^3$

$M - K \approx S^2 \times (0, \infty)$
asymptotic to a warped product
asymptotic to round cylinder after rescaling (more later)

"neck"



$M \approx S^2 \times \mathbb{R} \,/\, \Gamma$

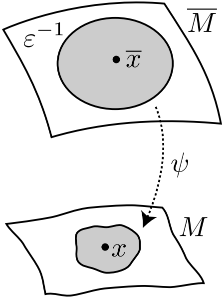
$g_t = -2t g_{S^2} + g_{\mathbb{R}}$

Brendle 2011

If $(M, (g_t))$ is also a steady soliton, then it is isometric to the Bryant soliton.

Singularity analysis, general case

(M, g) **Riemannian manifold,** $x \in M$ **point**



- curvature scale: $\rho(x) = (\frac{1}{3} R(x))^{-1/2}$
- $(\overline{M}, \overline{g}, \overline{x})$ pointed Riem. manifold (model space)
- $(\overline{M}, \overline{g}, \overline{x})$ local ε-model at x if there is

$$\psi : B(\overline{x}, 0, \varepsilon^{-1}) \longrightarrow M$$

such that $\psi(\overline{x}) = x$ and

$$\left\| \rho^{-2}(x) \psi^* g - \overline{g} \right\|_{C^{[\varepsilon^{-1}]}} < \varepsilon.$$

$(M, (g_t)_{t \in [0,T)})$ **Ricci flow**

- ... satisfies ε-canonical neighborhood assumption at scales $< r_0$ if all (x, t) with $\rho(x, t) < r_0$ are locally ε-modeled on the final time-slice $(\overline{M}, \overline{g}_0, \overline{x})$ of a pointed κ-solution.

Perelman 2002, precise form

$(M, (g_t)_{t \in [0,T)})$ satisfies the ε-canonical nbhd assumption at scales $< r(\varepsilon)$.

Ricci flow with surgery

Ricci flow with surgery:

$$(M_1, g^1_t), t \in [0, T_1],$$
$$(M_2, g^2_t), t \in [T_1, T_2],$$
$$(M_3, g^3_t), t \in [T_1, T_2], \quad \ldots$$

surgery

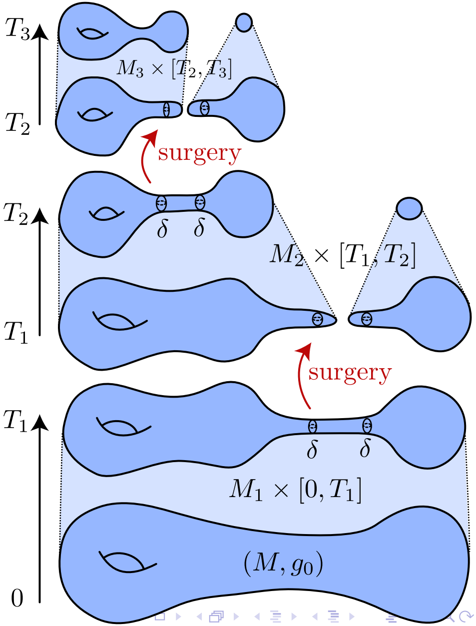
$$U_i^- \subset M_i \qquad U_i^+ \subset M_{i+1}$$

$$\varphi_i : (U_i^-, g^i_{T_i}) \xrightarrow{\cong} (U_i^+, g^{i+1}_{T_i})$$

surgery scale $\approx \delta \ll 1$

Theorem (Perelman 2003)

This process can be continued indefinitely.
No accumulation of T_i.

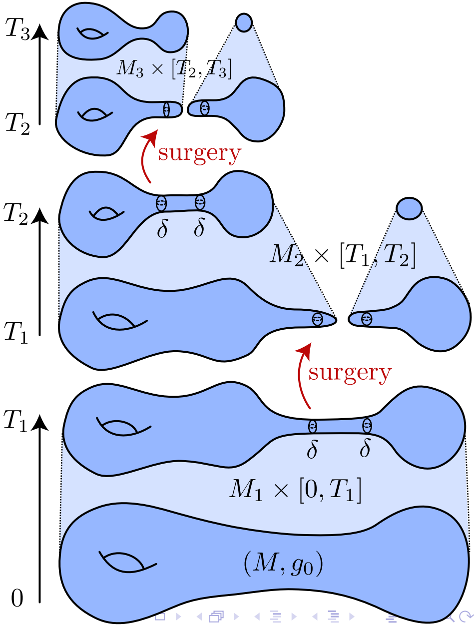


Note:

surgery process is not canonical
(depends on surgery parameters)

Perelman:

- *It is likely that [. . .] one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that.*
- *Our approach [. . .] is aimed at eventually constructing a canonical Ricci flow, [. . .] - a goal, that has not been achieved yet in the present work.*



Space-time picture

- Space-time 4-manifold:

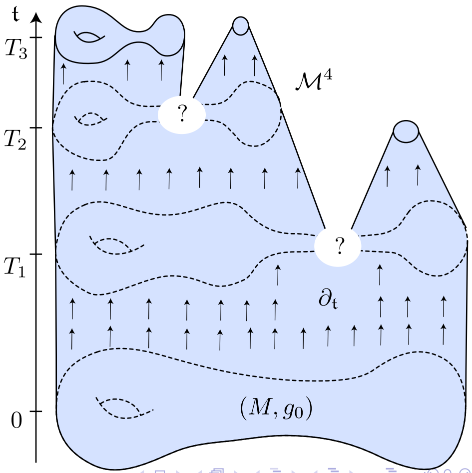
$$\mathcal{M}^4 = \big(M_1 \times [0, T_1] \ \cup_{\varphi_1} \ M_2 \times [T_1, T_2] \ \cup_{\varphi_2} \ M_3 \times [T_2, T_3] \ \cup_{\varphi_3} \ \ldots\big) - \mathcal{S}$$

$$\mathcal{S} = (M_1 \times \{T_1\} - U_1^-) \cup (M_2 \times \{T_1\} - U_1^+) \cup \ldots \quad \text{(surgery points)}$$

- Time function: $\mathfrak{t} : \mathcal{M} \to [0, \infty)$.
- Time-slice: $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$
- Time vector field: $\partial_\mathfrak{t}$ on $\mathcal{M}$ (with $\partial_\mathfrak{t} \cdot \mathfrak{t} = 1$).
- Metric g: on the distribution $\{d\mathfrak{t} = 0\} \subset T\mathcal{M}$
- Ricci flow equation: $\mathcal{L}_{\partial_\mathfrak{t}} g = -2 \operatorname{Ric}_g$

$(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g$ is called a Ricci flow spacetime.

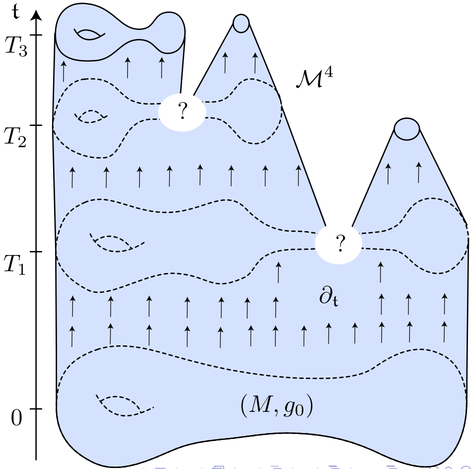
Note: there are "holes" at scale $\approx \delta$
space-time is δ-complete



Space-time picture

The spacetime $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$

- …satisfies the ε-canonical neighborhood assumption at scales $(C\delta, r_\varepsilon)$
- …is $C\delta$-complete
  i.e. $\{\rho \geq \rho', \mathfrak{t} \leq T\} \subset \mathcal{M}$
  is compact for all $\rho' > C\delta$, $T > 0$



Theorem (Kleiner, Lott 2014)

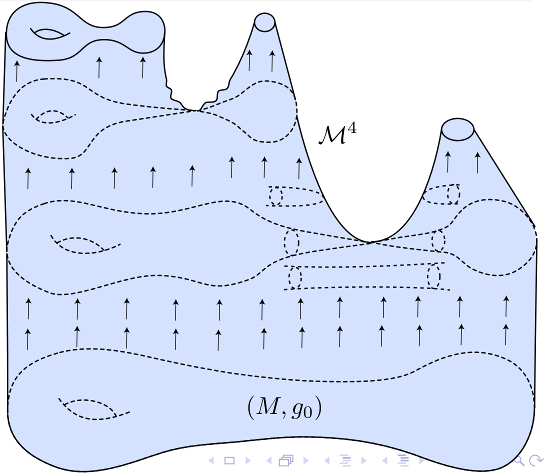
Given a compact (M^3, g_0), there is a Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ s.t.:

- initial time-slice: $(\mathcal{M}_0, g) = (M, g_0)$.
- $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ is 0-complete (i.e. "singularity scale $\delta = 0$")
- $\mathcal{M}$ satisfies the ε-canonical nbhd assumption at small scales for all $\varepsilon > 0$.

$(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ flows "through singularities at infinitesimal scale"

Remarks:

- g is smooth everywhere and not defined at singularities
- singular times may accummulate
- $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ arises as limit for $\delta_i \to 0$.



Statement of main results

Uniqueness

Theorem (Ba., Kleiner, 2017)

There is a constant $\varepsilon_{\mathrm{can}} > 0$ such that:
Every Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is uniquely determined by its initial time-slice $(\mathcal{M}_0, g_0)$, provided that it

- is 0-complete and
- satisfies the $\varepsilon_{\mathrm{can}}$-canonical neighborhood assumption below some positive scale.

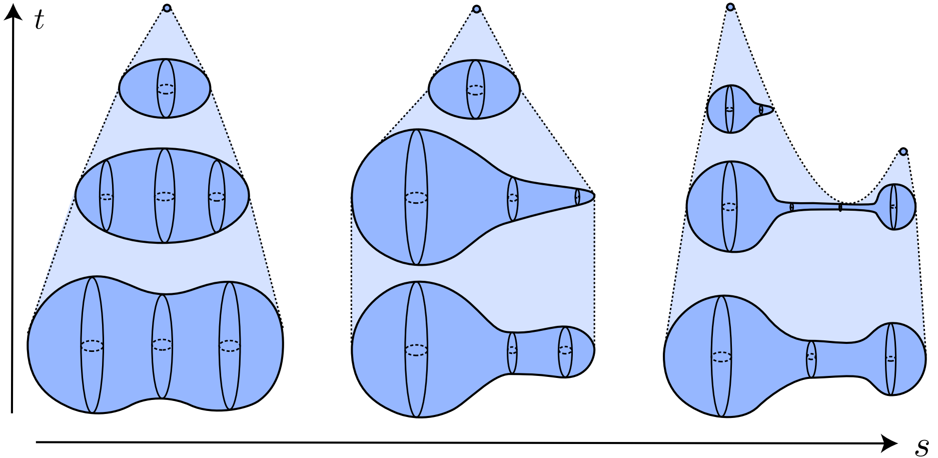
Corollary

For every compact (M^3, g_0) there is a unique, canonical singular Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ with $\mathcal{M}_0 = (M^3, g_0)$.

Continuity of RF space-times

continuous family of metrics $(g_s)_{s\in[0,1]}$ on M

$\rightsquigarrow \quad \{\mathcal{M}_s\}_{s\in[0,1]}$ canonical RF space-times



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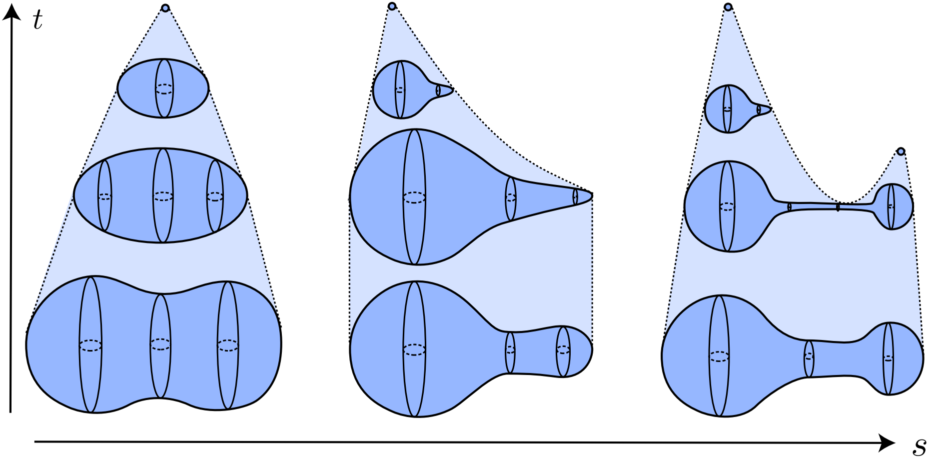

Applications

Corollary (Ba., Kleiner)

Every continuous/smooth family $(g_s)_{s\in\Omega}$ of Riemannian metrics on a compact manifold M^3 gives rise to a "continuous/smooth family of Ricci flow space-times".

Generalized Smale Conjecture

$\mathrm{Isom}(S^3/\Gamma) \longrightarrow \mathrm{Diff}(S^3/\Gamma)$ is a homotopy equivalence.

Hatcher 1983: Case $\Gamma = 1, \ldots$

Conjecture

$\mathcal{R}^+(S^3) = \{\text{metrics } g \text{ on } S^3 \mid R_g > 0\}$ is contractible.

Marques 2012: $\pi_0(\mathcal{R}^+(S^3)) = 0$.

Uniqueness

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- is 0-complete and
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Ingredients of proof

1. Blow-up analysis of almost singular part (talks 1+2)
2. Linear stability theory (talk 2)
3. Spatial Extension Principle (talk 3)

More blow-up analysis

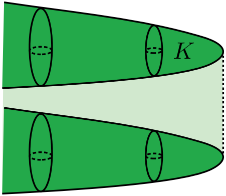
&

Toy Case

Recap κ-solutions

Recall:

- κ-solutions model high curvature / low ρ regions
- κ-solutions $\approx \mathbb{R}^3$ are not completely classified
- However, any κ-solution that is also a steady soliton is isometric to the Bryant soliton (M_{Bry}, g_{Bry}) modulo rescaling.



Hamilton 1993 + Brendle 2011

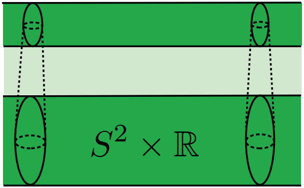
- $\partial_t R \geq 0$ on every κ-solution $(\overline{M}, (\overline{g}_t))$
- If $\partial_t R(x,t) = 0$ for some $(x,t) \in \overline{M} \times (-\infty, 0]$, then $(\overline{M}, (\overline{g}_t))$ is isometric to the Bryant soliton modulo rescaling.

(Strong) ε-necks

Recall: $\rho = (\frac{1}{3}R)^{-1/2}$

shrinking cylinder: $g_{S^2\times\mathbb{R},t} = (\frac{2}{3} - 2t)g_{S^2} + g_{\mathbb{R}}$

$$\rho(\cdot,0) \equiv 1, \qquad \rho(\cdot,-1) \equiv 2$$



(M,g) Riemannian manifold

ε-neck (at scale r): $U \subset M$ such that there is a diffeomorphism $\psi : S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to U$ with

$$\left\| r^{-2}\psi^* g - g_{S^2\times\mathbb{R},0} \right\|_{C^{[\varepsilon^{-1}]}(S^2\times(-\varepsilon^{-1},\varepsilon^{-1}))} < \varepsilon$$

$\psi(S^2 \times \{0\})$ central 2-sphere, consisting of centers of U

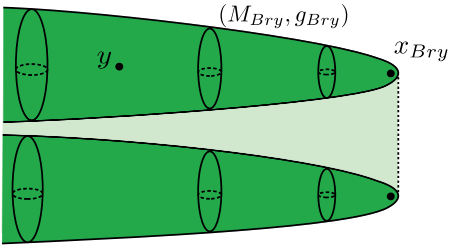
$(M,(g_t))$ Ricci flow

strong ε-neck (at scale r): as before, but

$$\left\| r^{-2}\psi^* g_{r^2 t+t_0} - g_{S^2\times\mathbb{R},t} \right\|_{C^{[\varepsilon^{-1}]}(S^2\times(-\varepsilon^{-1},\varepsilon^{-1})\times[-1,0])} < \varepsilon$$

(Strong) ε-necks

Example: Bryant soliton $(M_{Bry}, g_{Bry}, x_{Bry})$

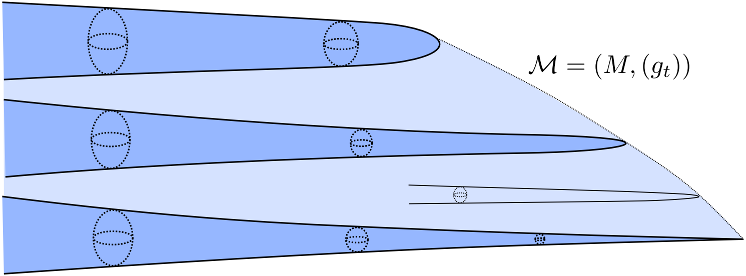


If $d_t(y, x_{Bry}) > C_{Bry}(\varepsilon)$, then (y, t) is a center of a strong ε-neck.

In a general RF, if (y, t) is a center of an $\varepsilon'(\varepsilon)$-neck, then it is also a center of a strong ε-neck.

Toy Case

We will only consider Ricci flow spacetimes $\mathcal{M}$ with the following properties:

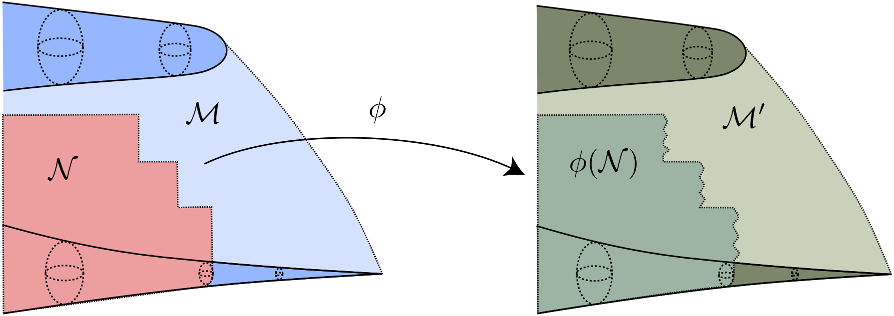


1. $\mathcal{M}_{(0,T)}$ comes from a non-singular, conventional RF $(M, (g_t)_{t \in (0,T)})$ that is κ-noncollapsed at scales < 1.
2. $\mathcal{M}_0$ can be compactified by adding a single point. $\mathcal{M}_0 \cong (M - X, g_0 := \lim_{t \searrow 0} g_t)$ for some $X \subset M$.
3. $\rho_{\min}(t) := \min_M \rho(\cdot, t) = \rho(x_t, t)$ is weakly increasing.
4. Every $(x, t) \in M \times (0, T)$ with $\rho(x, t) < 1$ is one of the following:
   - a center of a strong ε-neck at scale $\rho(x, t)$
   - locally ε-modeled on (M_{Bry}, g_{Bry}, y) for some $y \in M_{Bry}$ with $d_t(y, x_{Bry}) < C_{Bry}(\varepsilon)$.

(ε will be chosen small in the course of the talks.)

Strategy

- Let $\mathcal{M}$, $\mathcal{M}'$ be as before and suppose $\mathcal{M}_0 \cong \mathcal{M}'_0$.
- If we could show that $\mathcal{M}_t \cong \mathcal{M}'_t$ for some $t > 0$, then $\mathcal{M} \cong \mathcal{M}'$.
- Construct $\mathcal{N} \subset \mathcal{M}$ (comparison domain) and
  $$\phi : \mathcal{N} \to \mathcal{M}' \text{ (comparison map) that is } (1+\eta)\text{-bilipschitz}$$
- Let $\eta \to 0$ and $\mathcal{N} \to \mathcal{M}$, limit $\rightsquigarrow$ isometry between $\mathcal{M}, \mathcal{M}'$



Uniqueness in the non-singular case

$$(M, g(0)) \cong (M', g'(0)) \qquad \Longrightarrow \qquad (M, g(t)) \cong (M', g'(t))$$

Proof

- **Comparison map:** $\phi : M \longrightarrow M'$ such that $\phi^* g'(0) = g(0)$
- **Perturbation:** $h(t) = \phi^* g'(t) - g(t), \qquad h(0) \equiv 0$
- **DeTurck's trick:** If $\phi^{-1}(t)$ moves by harmonic map heat flow, then

$$\partial_t h = \triangle_{g(t)} h + 2\,\mathrm{Rm}_{g(t)}(h) + \nabla h * \nabla h + h * \nabla^2 h$$

(Ricci-DeTurck flow)

- Standard parabolic theory: $h(0) \equiv 0 \quad \Longrightarrow \quad h(t) \equiv 0$ **q.e.d.**

Later:

If $|h(t)| < \eta_{\mathrm{lin}} \ll 1$, then

$$\partial_t h \approx \triangle_{g(t)} h + 2\,\mathrm{Rm}_{g(t)}(h) \qquad \text{where} \qquad (\mathrm{Rm}(h))_{ij} = R_{istj} h_{st}$$

(linearized Ricci-DeTurck flow)

Linearized DeTurck flow

(g_t) Ricci flow, (h_t) linearized Ricci DeTurck flow

Anderson, Chow (2005)

$$\partial_t \frac{|h|}{R} \leq \triangle \frac{|h|}{R} - 2\frac{\nabla R}{R} \cdot \nabla \frac{|h|}{R}$$

Proof: $\square = \partial_t - \triangle$

$$\begin{aligned} \square \frac{|h|}{R} &= \frac{1}{R^2}\big(\square |h| \cdot R - |h| \square R\big) - 2\frac{\nabla R}{R} \cdot \nabla \frac{|h|}{R} \\ &\leq \frac{1}{R^2}\left(\frac{2\,\mathrm{Rm}(h,h)}{|h|} \cdot R - |h| \cdot 2|\mathrm{Ric}| \right) - 2\frac{\nabla R}{R} \cdot \nabla \frac{|h|}{R} \end{aligned}$$

Need

$$\mathrm{Rm}(h,h) R \leq |h|^2 |\mathrm{Ric}|^2$$

Can be checked using an “elementary” computation.

Vanishing Theorem

Let $(M,(g_t)_{t\in(-\infty,0]}$ be a κ-solution (e.g. shrinking cyliner, Bryant soliton) and $(h_t)_{t\in(-\infty,0]}$ a linearized RDTF such that

$$|h| \le CR^{1+\gamma}, \qquad \gamma > 0.$$

Then $h \equiv 0$.

Proof: $|h| \le CR^{1+\gamma} \le CC'R$. Choose $(x_i,t_i) \in M \times (-\infty,0]$ s.t.

$$\frac{|h|}{R}(x_i,t_i) \xrightarrow[i\to\infty]{} \sup_{M\times(-\infty,0]} \frac{|h|}{R} =: C_0.$$

$$R^\gamma(x_i,t_i) = \frac{R^{1+\gamma}(x_i,t_i)}{R(x_i,t_i)} \ge \frac{C^{-1}|h|}{R}(x_i,t_i) \xrightarrow[i\to\infty]{} C^{-1}C_0 > 0 \qquad (*)$$

After passing to a subsequence

$$(M,(g_{t+t_i})_{t\in(-\infty,0]},x_i) \xrightarrow[i\to\infty]{C^\infty-HCG} (M_\infty,(g_t^\infty)_{t\in(-\infty,0]},x_\infty)$$

$$(h_{t+t_i})_{t\in(-\infty,0]} \xrightarrow[i\to\infty]{} (h_t^\infty)_{t\in(-\infty,0]}$$

$|h^\infty| \le CR^{1+\gamma}$, $\quad \frac{|h^\infty|}{R} \le C_0 \quad$ (equality at $(x_\infty,0)$)

After passing to a subsequence

$$(M, (g_{t+t_i})_{t \in (-\infty,0]}, x_i) \xrightarrow[i \to \infty]{C^\infty - HCG} (M_\infty, (g_t^\infty)_{t \in (-\infty,0]}, x_\infty)$$

$$(h_{t+t_i})_{t \in (-\infty,0]} \xrightarrow[i \to \infty]{} (h_t^\infty)_{t \in (-\infty,0]}$$

$|h^\infty| \leq C R^{1+\gamma}$, $\quad \dfrac{|h^\infty|}{R} \leq C_0 \quad$ (equality at $(x_\infty, 0)$)

strong maximum principle $\implies \quad \dfrac{|h^\infty|}{R} = C_0$

$$R^\gamma = \frac{R^{1+\gamma}}{R} \geq \frac{C^{-1}|h^\infty|}{R} = C^{-1} C_0^{-1} > 0$$

on $M_\infty \times (-\infty, 0]$ q.e.d.

Linear $\longrightarrow$ Non-Linear

- (g_t) RF background
- (h_t) solution to non-linear RDTF

$$\partial_t h = \triangle h + 2\,\mathrm{Rm}(h) + \nabla h * \nabla h + h * \nabla^2 h$$

- **Observation:** Divide by $a > 0$

$$\partial_t \left(\frac{h}{a}\right) = \triangle\left(\frac{h}{a}\right) + 2\,\mathrm{Rm}\left(\frac{h}{a}\right) + a \cdot \nabla\left(\frac{h}{a}\right) * \nabla\left(\frac{h}{a}\right) + a \cdot \left(\frac{h}{a}\right) * \nabla^2\left(\frac{h}{a}\right)$$

- If $a_i \to 0$ and $\dfrac{h_i}{a_i} \to h_\infty$, then (assuming certain derivative bounds)

$$\partial_t h_\infty = \triangle h_\infty + 2\,\mathrm{Rm}(h_\infty)$$

(linearized RDTF)

$(g_t)_{t\in[0,T)}$ κ-noncollapsed RF, ε-canonical nbhd assumption,
$(h_t)_{t\in[0,T)}$ (non-linear) RDTF

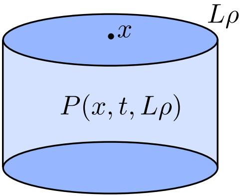
$$Q := e^{-Ht}\frac{|h|}{R^E}, \qquad E > 1$$

Given $\alpha > 0$ there are $H, \varepsilon, \eta, L > 0$ s.t. if

$$|h| \leq \eta \qquad on \qquad P = P(x, t, L\rho(x,t)),$$

then

$$Q(x,t) \leq \alpha \sup_P Q.$$



Proof: Assume not for E, α, κ. Choose $H_i, L_i \to \infty$, $\eta_i, \varepsilon_i \to 0$.
Counterexamples $(M_i, (g^i_t))$, (h_i), $|h_i| \leq \eta_i \to 0$, $r_i := \rho(x_i, t_i)$

$$Q(x_i,t_i) \geq \alpha \sup_{P_i} Q \quad \Longrightarrow \quad |h_i|(y,t) \leq \alpha^{-1} e^{-H_i(t_i - t)} \frac{R^E(y,t)}{R^E(x_i,t_i)} \cdot |h_i|(x_i,t_i) \quad \text{on} \quad P_i$$

$$(M_i, (r_i^{-2} g_{r_i^2 t + t_i}), x_i) \xrightarrow[i\to\infty]{C^\infty - HCG} (M_\infty, (g^\infty_t)_{t\in(-\infty,0]}, x_\infty)$$

$$\frac{h_i}{|h_i|(x_i,t_i)} \xrightarrow[i\to\infty]{} (h_{\infty,t})_{t\in(-\infty,0]} \qquad \text{(linearized RDTF)}$$

$$Q(x_i,t_i) \geq \alpha \sup_{P_i} Q \quad \Longrightarrow \quad |h_i|(y,t) \leq \alpha^{-1} e^{-H_i(t_i-t)} \frac{R^E(y,t)}{R^E(x_i,t_i)} \cdot |h_i|(x_i,t_i) \quad \text{on} \quad P_i$$

$$(M_i, (r_i^{-2} g_{r_i^2 t + t_i}), x_i) \xrightarrow[i\to\infty]{C^\infty-HCG} (M_\infty, (g_t^\infty)_{t\in(-\infty,0]}, x_\infty)$$

$$\frac{h_i}{|h_i|(x_i,t_i)} \xrightarrow[i\to\infty]{} (h_{\infty,t})_{t\in(-\infty,0]} \qquad \text{(linearized RDTF)}$$

Then $|h_\infty|(x_\infty, 0) = 1$

$$|h_\infty|(y,t) \leq \lim_{i\to\infty} e^{H_i r_i^2 \cdot t} R^E(y,t)$$

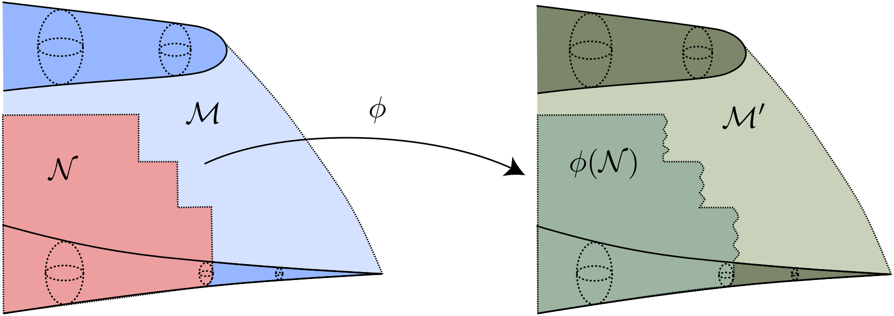
Case $\liminf_{i\to\infty} \mathbf{r_i} > \mathbf{0}$: $|h_\infty|(\cdot,t) \equiv 0$ for $t < 0$ $\Longrightarrow$ $h_\infty \equiv 0$

Case $\liminf_{i\to\infty} \mathbf{r_i} = \mathbf{0}$: imit is κ-solution
$|h_\infty| \leq \alpha^{-1} R^E$ $\Longrightarrow$ $h_\infty \equiv 0$

q.e.d.

Recap: Strategy

- Let $\mathcal{M}$, $\mathcal{M}'$ be as before and suppose $\mathcal{M}_0 \cong \mathcal{M}'_0$.
- Construct $\mathcal{N} \subset \mathcal{M}$ (comparison domain) and
  $\phi : \mathcal{N} \to \mathcal{M}'$ (comparison map) that is $(1+\eta)$-bilipschitz
- Let $\eta \to 0$ and $\mathcal{N} \to \mathcal{M}$, limit $\rightsquigarrow$ isometry between $\mathcal{M}, \mathcal{M}'$



Construction of the comparison domain

- $\lambda = \lambda(\delta_n)$ to be determined later
- Choose $r_{\text{comp}} \ll 1$ (comparison scale)
- $t_j := j \cdot r_{\text{comp}}^2$
- $t_J = T$
- choose j_0 minimal s.t. $\rho_{\min}(t_{j_0}) \geq \lambda r_{\text{comp}}$

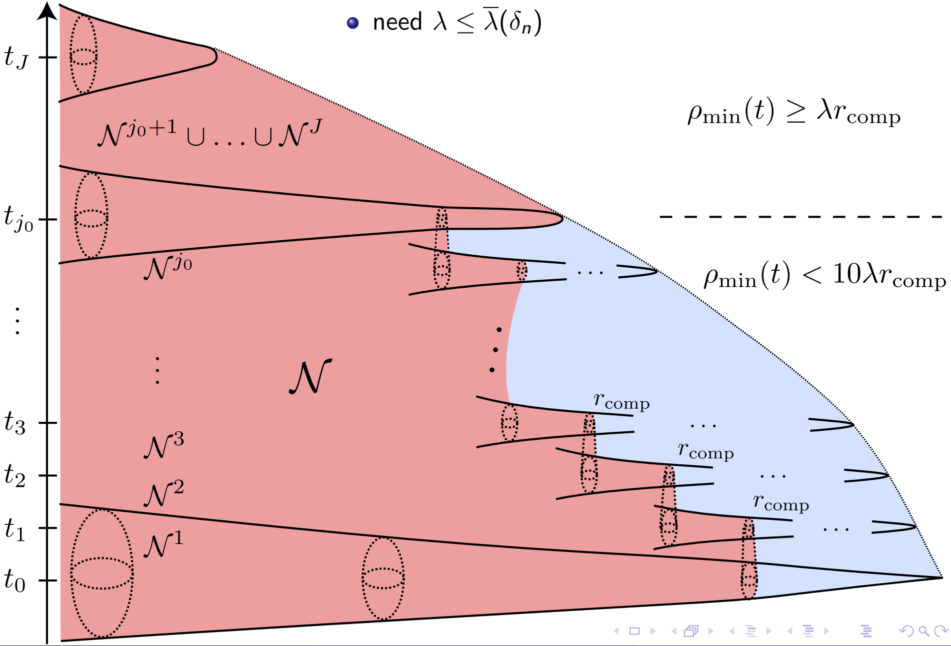
There is a comparison domain

$$\mathcal{N} = (\mathcal{N}^1 \cup \ldots \cup \mathcal{N}^{j_0}) \cup (\mathcal{N}^{j_0+1} \cup \ldots \cup \mathcal{N}^J) \quad \subset \quad \mathcal{M}$$

such that

- $\mathcal{N}^j = N_j \times [t_{j-1}, t_j]$
- $\partial \mathcal{N}^j_{t_j}$ is central 2-sphere of strong δ_n-neck at scale r_{comp}
- $\rho > \frac{1}{2} r_{\text{comp}}$ on $\mathcal{N}^1 \cup \ldots \cup \mathcal{N}^{j_0}$
- $\mathcal{N}^j = M \times [t_{j-1}, t_j]$ for $j \geq j_0 + 1$

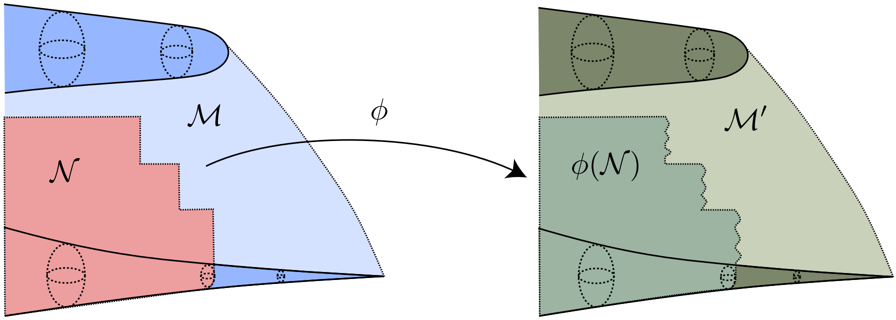
- $\partial\mathcal{N}^j$ central 2-spheres of strong δ_n-necks
- need $\lambda \leq \overline{\lambda}(\delta_n)$



Construction of comparison map for $t \leq t_{j_0}$

Goal: Construct comparison map $\phi : \mathcal{N}^1 \cup \ldots \cup \mathcal{N}^{j_0} \to \mathcal{M}'$, evolving by harmonic map heat flow s.t.

$$|h| = |\phi^* g' - g| < \eta$$

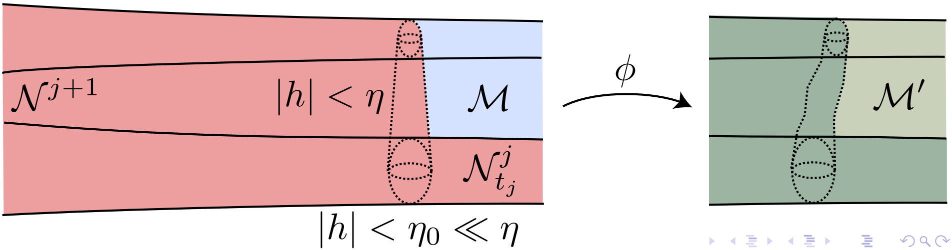


Strategy:

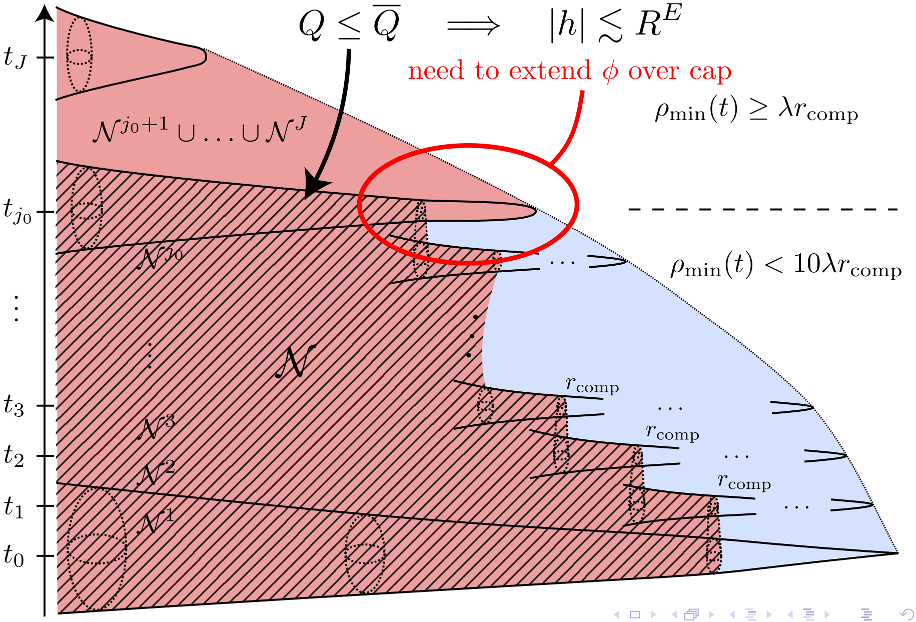
- $\phi|_{\mathcal{N}_0}$ given since $\mathcal{M}_0 \cong \mathcal{M}'_0$
- For each $j = 0, \ldots, j_0 - 1$ solve HMHF with initial data $\phi|_{\mathcal{N}^j_{t_j}}$

  $\rightsquigarrow \qquad \phi|_{\mathcal{N}^{j+1}} : \mathcal{N}^{j+1} \to \mathcal{M}'$.
- This works if $\delta_n < \overline{\delta}_n(\eta)$ and $|h| < \eta_0 \ll \eta$ near $\partial\mathcal{N}^{j+1}_{t_j}$
- Choose $\overline{Q}$ such that $Q < \overline{Q}$ implies $|h| < \eta$. Ensure $Q < \overline{Q}$ near $\partial\mathcal{N}^{j+1}_{t_j}$.
- Interior decay estimate: If $d_{t_j}(\partial\mathcal{N}^{j+1}_{t_j}, \partial\mathcal{N}^{j}_{t_j}) > L r_{\mathrm{comp}}$, then

$$Q < \alpha\overline{Q}, \qquad \alpha \ll 1 \qquad \Longrightarrow \qquad |h| < \eta_0 \qquad \text{near} \qquad \partial\mathcal{N}^{j+1}_{t_j}$$



Construction of comparison map at $t = t_{j_0}$

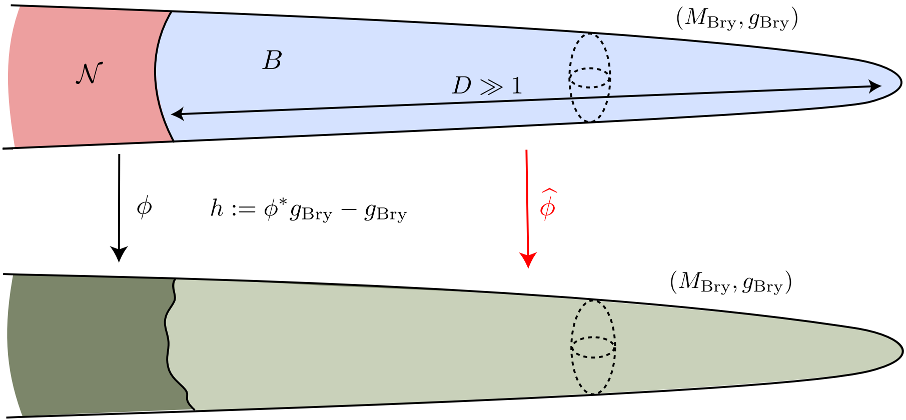


Cantilever Paradox

Where do you feel safer?

long cantilever,
good engineering

short cantilever,
sketchy engineering



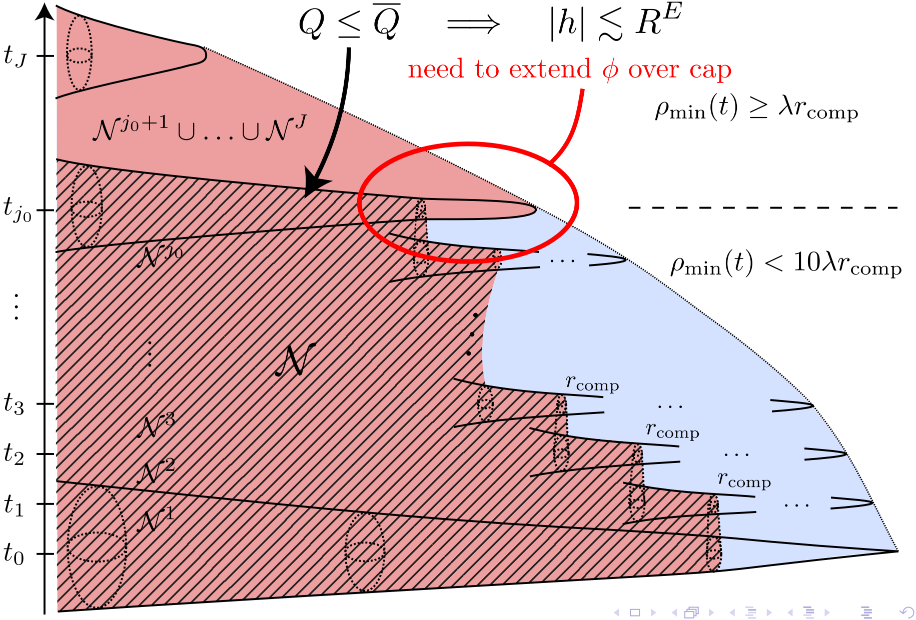
Assume that $Q = e^{Ht} \dfrac{|h|}{R^E} < \overline{Q}$ on $\mathcal{N}$.

Bryant Extension Principle ("Cap Extension")

If $E > 100$ and $D \geq D_0(\eta)$, then there is an "extension" $\widehat{\phi} : B \cup \mathcal{N} \to M_{\mathrm{Bry}}$ of $\phi : \mathcal{N} \to M_{\mathrm{Bry}}$ such that

$$\left| \widehat{\phi}^* g_{\mathrm{Bry}} - g_{\mathrm{Bry}} \right| < \eta$$

Proof, concluded



Proof, concluded

- Extend $\phi|_{\mathcal{N}^{j_0}_{t_{j_0}}}$ onto $\mathcal{N}^{j_0+1}_{t_{j_0}}$ via Bryant Extension Principe (at time t_{j_0})
- Extend $\phi|_{\mathcal{N}^{j_0+1}_{t_{j_0}}}$ onto $\mathcal{N}^{j_0+1} \cup \ldots \cup \mathcal{N}^J$ by solving HMHF (no boundary!).
- Control $|h|$ using $Q^* = e^{Ht} \dfrac{|h|}{R^{E^*}}$ for $E^* \ll E$.

  (Bryant Extension Principle $\Longrightarrow$ $Q^* \leq \overline{Q}^*$ at time t_{j_0}.)
- q.e.d.

The general case

Problems that may arise

- Caps may not always be modeled on Bryant solitons
  Solution: Perform cap extension "at the right time",
  when $\partial_t R \leq 0$ on $\mathcal{M}$ and $\mathcal{M}'$
  $\rightsquigarrow$ $\mathcal{N}$ and ϕ must be chosen simultaneously

- There may be more than one cusp
  Solution: Continue the neck and cap extension process after the first cap extension.

- Curvature of the cap may increase again after cap extension (and there may be an accumulation of singular times)
  Solution: Perform a "cap removal" when the curvature exceeds a certain threshold.
  Ensure that "cap extensions" and "cap removals" are sufficiently separated in space and time such that Q has time to "recover" after a cap extension.