Uniqueness of Weak Solutions to the Ricci Flow and Topological Applications

Richard H Bamler
(joint work with Bruce Kleiner, NYU)

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Structure of Talk

- Part I: Topological Results
- Part II: Ricci flow, Weak solutions, Uniqueness,
  Continuous dependence
- Part III: Applications to Topology

Part I: Topological Results

Basic definitions

M (mostly) 3-dimensional, compact, orientable manifold

Recall: The topology of 3-manifolds is sufficiently well understood due to the resolution of the Poincaré and Geometrization Conjectures by Perelman, using Ricci flow.

Main objects of study:

- $\mathrm{Met}(M)$: space of Riemannian metrics on M
- $\mathrm{Met}_{PSC}(M) \subset \mathrm{Met}(M)$: subset of metrics with positive scalar curvature
- $\mathrm{Diff}(M)$: space of diffeomorphisms $\phi : M \to M$

. . . each equipped with the C^∞-topology.

Goal: Classify these spaces up to homotopy (using Ricci flow)!

$\mathrm{Met}(M)$ is contractible

Space of PSC-metrics

Main Result 1:

Ba., Kleiner 2019

$\mathrm{Met}_{PSC}(M)$ is either contractible or empty.

History:

- true in dimension 2 (via Uniformization Theorem or Ricci flow (see later))
- Hitchin 1974; Gromov, Lawson 1984; Botvinnik, Hanke, Schick, Walsh 2010: Further examples with $\pi_i(\mathrm{Met}_{PSC}(M^n)) \neq 1$ for certain (large) i, n.
- Marques 2011 (using Ricci flow with surgery):
  $\mathrm{Met}_{PSC}(M^3)/\operatorname{Diff}(M^3)$ is path-connected
  $\mathrm{Met}_{PSC}(S^3)$ is path-connected,

Diffeomorphism groups

Smale 1958: $O(3) \simeq \mathrm{Diff}(S^2)$

Smale Conjecture: $O(4) \simeq \mathrm{Diff}(S^3)$
proven by Hatcher in 1983

For a general spherical space form $M = S^3/\Gamma$ consider the inclusion map

$$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$$

Generalized Smale Conjecture

This map is a homotopy equivalence.

- Verified for a handful of other spherical space forms, but open e.g. for $\mathbb{R}P^3$.
- All proofs so far are purely topological and technical. No uniform treatment.

Main Result 2:

Theorem (Ba., Kleiner 2019)

The Generalized Smale Conjecture is true.

Remarks:

- Proof via Ricci flow (first purely topological application of Ricci flow since Perelman's work ~ 15 years ago).
- Uniform treatment of all cases.
- Alternative proof in the S^3-case (Smale Conjecture).
- There are two proofs:
  - "Short" proof (Ba., Kleiner 2017): GSC if $M \not\approx S^3, \mathbb{R}P^3$, M hyperbolic
  - Long proof (Ba., Kleiner 2019): full GSC and $S^2 \times \mathbb{R}$-cases

Similar techniques imply results in non-spherical case:

- If M is closed and hyperbolic, then $\mathrm{Isom}(M) \simeq \mathrm{Diff}(M)$.
  (topological proof by Gabai 2001)
- If (M, g) is aspherical and geometric and g has maximal symmetry, then $\mathrm{Isom}(M) \simeq \mathrm{Diff}(M)$.
  (new in non-Haken infranil case.)
- $\mathrm{Diff}(S^2 \times S^1) \simeq O(2) \times O(3) \times \Omega O(3)$
  (topological proof by Hatcher)
- $\mathrm{Diff}(\mathbb{R}P^3 \# \mathbb{R}P^3) \simeq O(1) \times O(3)$
  (topological proof by Hatcher)

Connection to Ricci flow

Lemma

For any $g \in \mathrm{Met}_{K\equiv\pm 1}(M)$:

$$\mathrm{Isom}(M,g) \simeq \mathrm{Diff}(M) \qquad \Longleftrightarrow \qquad \mathrm{Met}_{K\equiv\pm 1}(M) \ \text{ contractible}$$

Proof: Fiber bundle

$$\begin{aligned} \mathrm{Isom}(M,g) \longrightarrow \mathrm{Diff}(M) &\longrightarrow \mathrm{Met}_{K\equiv\pm 1}(M) \\ \phi &\longmapsto \phi^* g \end{aligned}$$

Apply long exact homotopy sequence.

This reduces both results to:

Theorem (Ba., Kleiner 2019)

$\mathrm{Met}_{PSC}(M)$ and $\mathrm{Met}_{K\equiv 1}(M)$ are each either contractible or empty.

or equivalently:

$$\pi_k(\mathrm{Met}(M), \mathrm{Met}_{PSC/K\equiv 1}(M)) = 1.$$

Part II: Ricci flow, Weak solutions, Uniqueness, Continuous dependence

Ricci flow: $(M, g(t)),\ t \in [0, T)$

$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}, \qquad g(0) = g_0 \qquad (*)$$

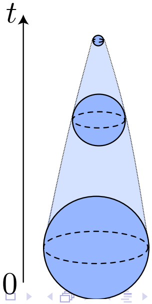
Short-time existence (Hamilton):

- For every initial condition g_0 the initial value problem $(*)$ has a unique solution for maximal $T \in (0, \infty]$.
- If $T < \infty$, then "singularity at time T". Curvature $|\mathrm{Rm}|$ blows up as $t \nearrow T$.

Example: Round shrinking sphere

$M = S^n$

$g(t) = (1 - 2(n-1)t) g_{S^n}$.

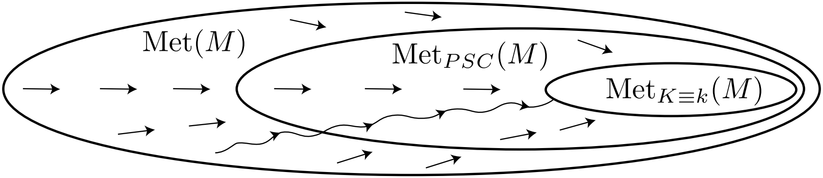


Ricci flow in 2D

Hamilton, Chow: On S^2 for any initial condition g_0 we have

$$T = \frac{\mathrm{vol}(S^2, g_0)}{8\pi}, \qquad (T-t)^{-1} g(t) \longrightarrow g_{\mathrm{round}}$$

Interpretation on the space of metrics:



- Preservation of positive scalar curvature (in all dimensions)
- $\rightsquigarrow$ deformation retractions from $\mathrm{Met}(S^2)$ and $\mathrm{Met}_{PSC}(S^2)$ onto $\mathrm{Met}_{K\equiv 1}(S^2)$

Theorem

$\mathrm{Met}_{PSC}(S^2) \simeq \mathrm{Met}_{K\equiv 1}(S^2) \simeq \mathrm{Met}(S^2) \simeq *$
Therefore $\mathrm{Diff}(S^2) \simeq O(3)$.

Ricci flow in 3D

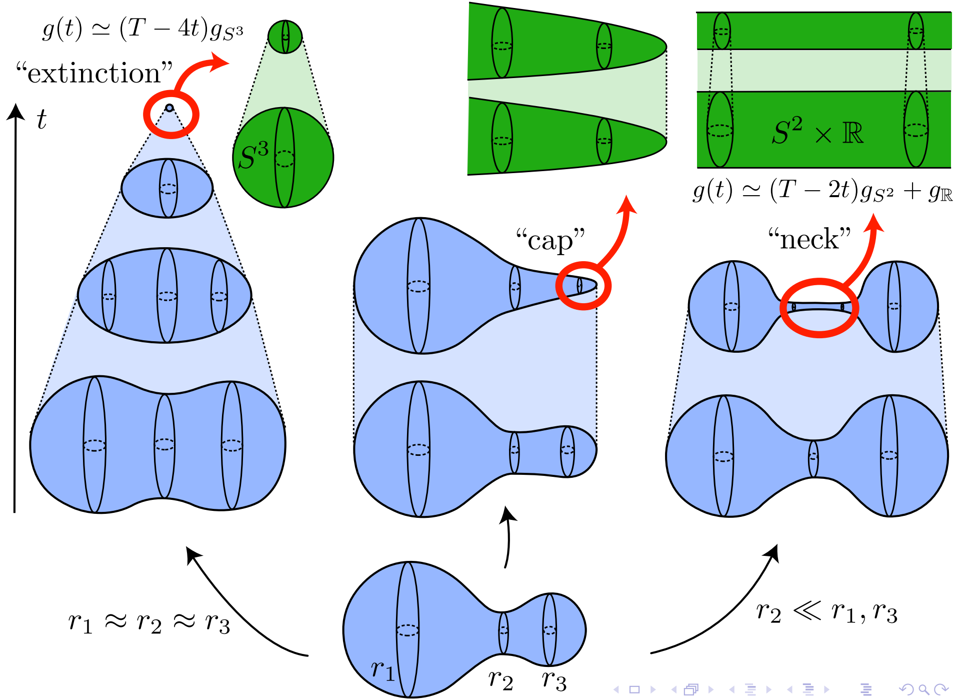
Difficulties:

- Flow may incur non-round and non-global singularities.
- Necessary to extend the flow past the first singular time (surgeries).
- Continuous dependence on initial data?

Results:

- Perelman: Qualitative classification of singularity models (κ-solutions)
- Brendle 2018 / Ba., Kleiner 2019: Further classification / rotational symmetry of κ-solutions

Example: rotationally symmetric dumbbell



$g(t) \simeq (T-4t)g_{S^3}$
"extinction"
t
S^3
"cap"
$S^2 \times \mathbb{R}$
$g(t) \simeq (T-2t)g_{S^2} + g_{\mathbb{R}}$
"neck"
$r_1 \approx r_2 \approx r_3$
$r_2 \ll r_1, r_3$
r_1
r_2
r_3

Ricci flow with surgery

Given (M, g_0) construct
Ricci flow with surgery:

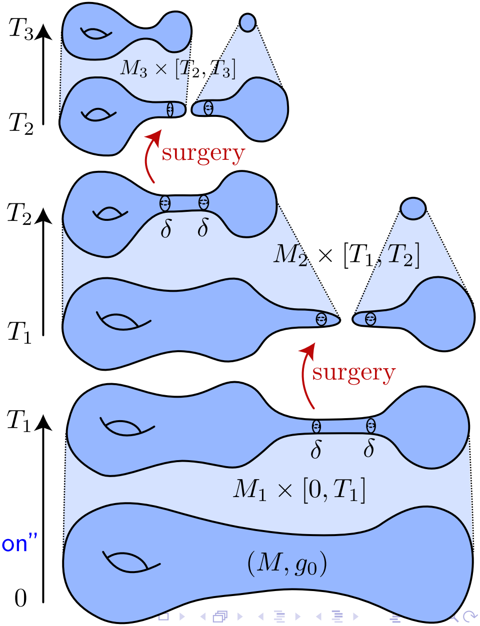
$$(M_1, g_1(t)), t \in [0, T_1],$$

$$(M_2, g_2(t)), t \in [T_1, T_2],$$

$$(M_3, g_3(t)), t \in [T_2, T_3], \quad \ldots$$

Observations:

- surgery scale $\approx \delta \ll 1$
- high curvature regions are ε-close to singularity models from before: "ε-canonical neighborhood assumption"

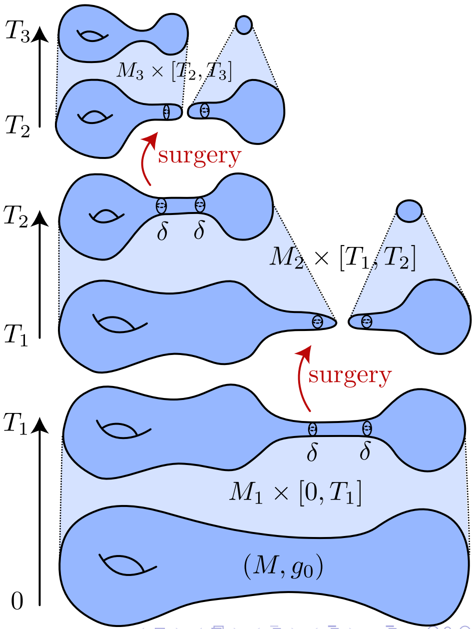


RF with surgery was used to prove Poincaré & Geometrization Conjectures

Drawback:
surgery process is not canonical (depends on surgery parameters)

Perelman:

- *It is likely that [. . .] one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that.*
- *Our approach [. . .] is aimed at eventually constructing a canonical Ricci flow, [. . .] - a goal, that has not been achieved yet in the present work.*



Theorem (Ba., Kleiner, Lott)

Perelman's "conjecture" is true:

- There is a notion of a weak Ricci flow "through singularities" and we have existence and uniqueness within this class.
- This weak flow is a limit of Ricci flows with surgery, where surgery scale $\delta \to 0$.

Comparison with Mean Curvature Flow:

- Notions of weak flows: Level Set Flow, Brakke flow
- General case: fattening $\cong$ non-uniqueness
- Mean convex case: non-fattening $\cong$ uniqueness
- 2-convex case: uniqueness + weak flow is limit of MCF with surgery as surgery scale $\delta \to 0$

How to take limits of sequences of Ricci flows with surgery?

Space-time 4-manifold:

$$\mathcal{M}^4 = \big(M_1 \times [0, T_1] \;\cup\; M_2 \times [T_1, T_2] \;\cup\; M_3 \times [T_2, T_3] \;\cup\; \ldots\big) - \text{surgery points}$$

Time function: $\mathfrak{t} : \mathcal{M} \to [0, \infty)$

Time-slices: $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$

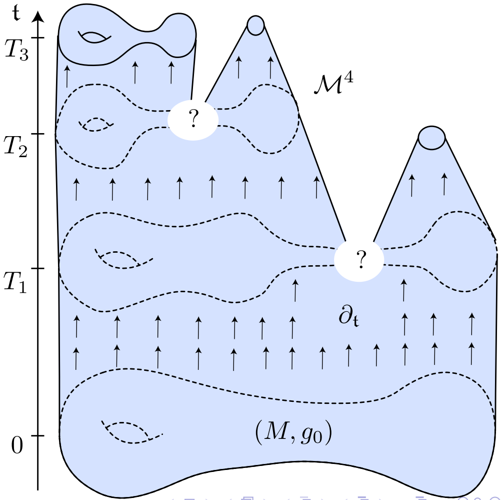
Time vector field:
$\partial_{\mathfrak{t}}$ on $\mathcal{M}$ (with $\partial_{\mathfrak{t}} \cdot \mathfrak{t} = 1$)

Metric g: on the distribution
$\ker d\mathfrak{t} \subset T\mathcal{M}$

Ricci flow equation:
$\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2 \operatorname{Ric}_g$

$\mathcal{M} = (\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is called a
Ricci flow spacetime.

Note: there are "holes" at scale $\approx \delta$
space-time is δ-complete



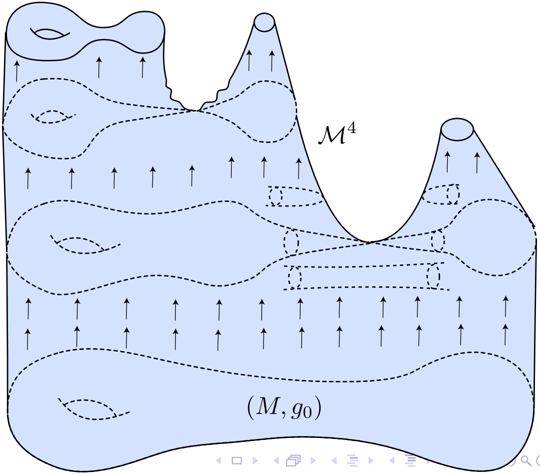
Kleiner, Lott 2014: Compactness theorem and $\delta_i \to 0$
$\Longrightarrow$ existence of singular Ricci flow starting from any (M, g)

Singular Ricci flow: Ricci flow spacetime $\mathcal{M}$ that:

- is 0-complete (i.e. "surgery scale $\delta = 0$")
- satisfies the ε-canonical neighborhood assumption for small ε.

Remarks:

- $\mathcal{M}$ is smooth everywhere and not defined at singularities
- singular times may accummulate



Uniqueness

Theorem (Ba., Kleiner 2016)

$\mathcal{M}$ is uniquely determined by its initial time-slice $(\mathcal{M}_0, g_0)$ up to isometry.

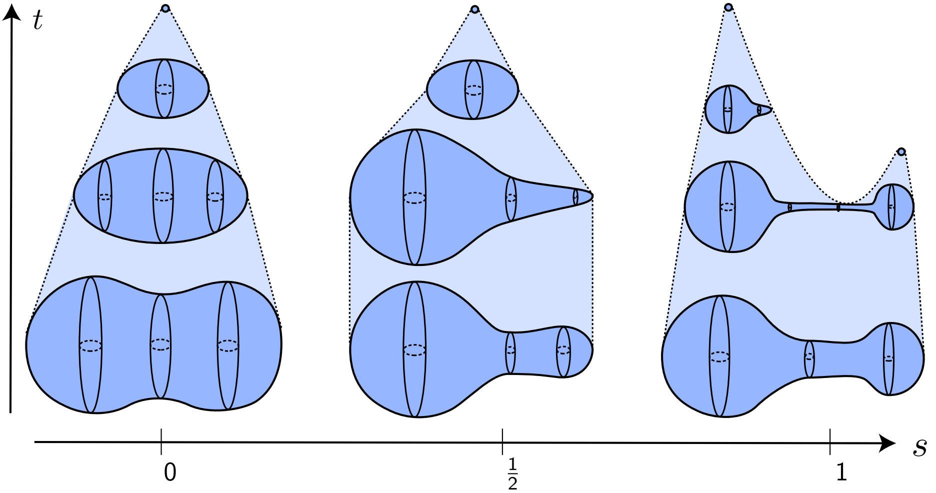
So for any (M, g) there is (up to isometry) a canonical singular Ricci flow $\mathcal{M}$ with initial time-slice $(\mathcal{M}_0, g_0) \cong (M, g)$.

Write: $\mathcal{M}^g$.

Uniqueness $\longrightarrow$ Continuous dependence

continuous family of metrics $(g^s)_{s\in[0,1]}$ on M

$\rightsquigarrow \quad \{\mathcal{M}^s := \mathcal{M}^{g^s}\}_{s\in[0,1]}$ singular RFs



Uniqueness $\longrightarrow$ Continuous dependence

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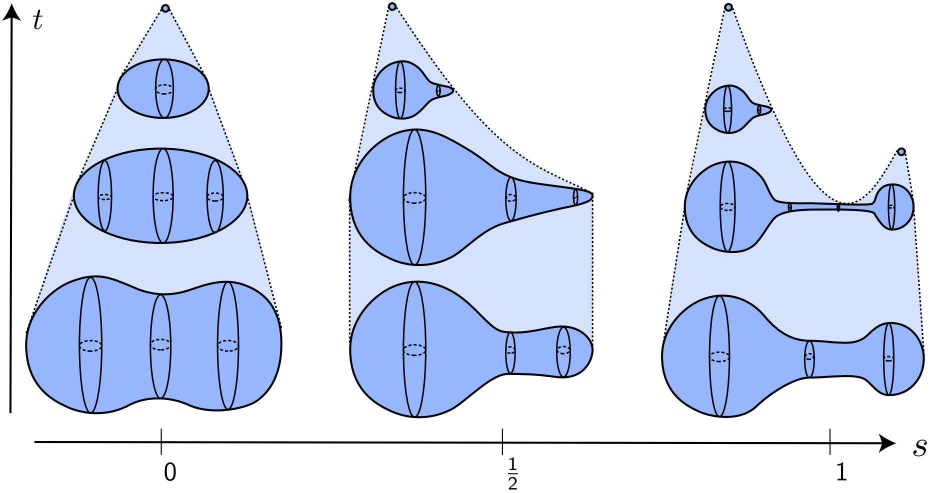

Continuity of singular RFs

$\mathcal{M}^g$ depends continuously on its initial metric g.

Precise statement:

Theorem (Ba., Kleiner 2019)

Given a continuous family $(g^s)_{s \in X}$ of Riemannian metrics on M over some topological space X, there is a continuous family of singular RFs $(\mathcal{M}^s = \mathcal{M}^{g^s})_{s \in X}$. That is:

- A topology on $\sqcup_{s \in X} \mathcal{M}^s$ such that the projection
$$\bigsqcup_{s \in X} \mathcal{M}^{g^s} \longrightarrow X$$
is a topological submersion.
- A compatible lamination structure on $\sqcup_{s \in X} \mathcal{M}^s$ with leaves $\mathcal{M}^s$ with respect to which all objects $\mathfrak{t}^s, \partial^s_{\mathfrak{t}}, g^s$ are transversely continuous.

Part III: Applications to Topology

Setup

Theorem (Ba., Kleiner 2019)

$\pi_k(\mathrm{Met}(M), \mathrm{Met}_{PSC/K\equiv 1}(M)) = 1$

Meaning (PSC-case): For any family of metrics $(h_{s,0})_{s\in D^k}$ on M, where

$$h_{s,0} \;\text{ has }\; PSC \qquad \text{for} \quad s \in \partial D^k,$$

there is a homotopy $(h_{s,t})_{s\in D^k\times[0,1]}$, s.t.

$$h_{s,1} \;\text{ has }\; PSC \qquad \text{for} \quad s \in D^k$$
$$h_{s,t} \;\text{ has }\; PSC \qquad \text{for} \quad s \in \partial D^k, \quad t \in [0,1]$$

Previous results: $(h_{s,0})_{s\in D^k} \rightsquigarrow$ cont. family of singular RFs $(\mathcal{M}^s = \mathcal{M}^{g^s})_{s\in D^k}$

Remaining Conversion Problem: Convert continuous family of sing. RFs $(\mathcal{M}^s)_{s\in X}$ with initial time-slice $\mathcal{M}^s_0 = M$ to $(h_{s,t})_{s\in X, t\in[0,1]}$ with:

1. $(M, h_{s,0}) \cong (\mathcal{M}^s_0, g^s_0)$.
2. $h_{s,1}$ has PSC.
3. If $\mathcal{M}^s$ has PSC, then so does $h_{s,t}$ for all $t \in [0,1]$.

Conversion Problem: Given $(\mathcal{M}^s)_{s \in X}$, find $(h_{s,t})_{s \in X, t \in [0,1]}$ s.t.:

1. $(M, h_{s,0}) \cong (\mathcal{M}^s_0, g^s_0)$
2. $h_{s,1}$ has PSC
3. If $\mathcal{M}^s$ has PSC, then so does $h_{s,t}$ for all $t \in [0,1]$.

- **Rounding procedure:** perturb metrics on $(\mathcal{M}^s)_{s \in X}$ so that they are round or rotationally symmetric in high curvature regions (this works because κ-solutions are round or rot. symmetric)
- **Strategy:** Construct $(h_{s,t})$ by backwards induction over time.
- **Problem:** For any fixed $T \geq 0$ the family $s \mapsto \mathcal{M}^s_T$ of time-T-slices is a "continuous family of Riemannian manifolds" whose topology may vary.
- **New notion:** "Partial homotopy at time T"

Conversion Problem: Given $(\mathcal{M}^s)_{s \in X}$, find $(h_{s,t})_{s \in X, t \in [0,1]}$ s.t.:

1. $(M, h_{s,0}) \cong (\mathcal{M}^s_0, g^s_0)$
2. $h_{s,1}$ has PSC
3. If $\mathcal{M}^s$ has PSC, then so does $h_{s,t}$ for all $t \in [0,1]$.

Partial homotopy at time T:
Notion involving families of metrics $(h_{s,t})$ as in 1-3, but with

$$\text{“}(M, h_{s,0}) \cong (\mathcal{M}^s_T, g^s_T)\text{”},$$

defined where $|\mathrm{Rm}| \lesssim r^{-2}$ over a simplicial decomposition of X.

Lemma

- If $T \gg 0$, then there is an (empty) partial homotopy at time T for $(\mathcal{M}^s)_{s \in X}$.
- A partial homotopy at time T for $(\mathcal{M}^s)_{s \in X}$, can be transformed into a partial homotopy at time $T - \varepsilon$, where $\varepsilon > 0$ is uniform, through certain modification moves.
- If there is a partial homotopy at time $T = 0$ for $(\mathcal{M}^s)_{s \in X}$, then there is a family $(h_{s,t})$ satisfying 1-3

Partial homotopy at time T:

Fix a simplicial decomposition of X. For each simplex $\sigma \subset X$ choose:

- a continuous family of compact domains $(Z^\sigma_s \subset \mathcal{M}^s_T)_{s\in\sigma}$. (roughly: $Z^\sigma_s \approx \{|\mathrm{Rm}| \lesssim r^{-2}_{\dim\sigma}\}$ for $r_0 \ll \ldots \ll r_n$.)
- a continuous family of Riemannian metrics $(h^\sigma_{s,t})_{s\in\sigma, t\in[0,1]}$ on (Z^σ_s).

such that ❶-❸ hold starting at time T and:

- **Compatibility:** If $s \in \tau \subset \sigma$, then $Z^\sigma_s \subset Z^\tau_s$ and $h^\sigma_{s,t} = h^\tau_{s,t}|_{Z^\sigma_s}$.
- **Largeness of the domains:** $|\mathrm{Rm}| \gtrsim r^{-2}_{\dim\sigma}$ on $\mathcal{M}^s_T \setminus Z^\sigma_s$ ($\Rightarrow$ round or rot. symmetric)
- **"Contractible ambiguity":** $h^\tau_{s,t}$ is round or rot. symmetric on any $Z^\tau_s \setminus Z^\sigma_s$.

Modification Moves:

- Reducing T to $T - \varepsilon$ if (Z^σ_s) stay away from high curvature regions.
- Passing to a simplicial refinement.
- Enlarging some (Z^σ_s) by a family of round or rot. symmetric subsets.
- Shrinking some (Z^σ_s) by removing a family of disks.