You know, for a mathematician, he did not have enough imagination. But he has become a poet and now he is fine.

Name and section: ______________________

1. (3 points) Determine the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$.

Solution: As usual, we'll use the ratio test. This gives

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}\sqrt{n}}{(x-1)^n \sqrt{n+1}} \right| = |x-1| \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = |x-1| \tag{1}$$

and the condition that $|x-1| < 1$ tells us that the radius of convergence is 1 and that the interval of convergence is centered at $x = 1$. That makes the endpoints $x = 0, x = 2$, which we still have to check. At $x = 0$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \tag{2}$$

which is an alternating series with $a_n = \frac{1}{\sqrt{n}}$ positive, decreasing, and going to 0, so it converges by the alternating series test. at $x = 2$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tag{3}$$

which diverges by the p-series test. Altogether, we find that the interval of convergence is $[0, 2)$.

2. (3 points) Find a power series representation of the function $f(x) = \dfrac{1}{1-x^2}$ and determine its radius of convergence.

Solution: There are several ways to do this problem. One is to use geometric series to write

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + ... \tag{4}$$

Another is to use partial fraction decomposition and write

$$\frac{1}{(1+x)(1-x)} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) \tag{5}$$

then use two geometric series to write

$$\frac{1}{(1+x)(1-x)} = \frac{1}{2}\left(\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(-1)^n x^n\right) \tag{6}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}(1+(-1)^n)x^n\right) \tag{7}$$

$$= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} x^n \tag{8}$$

which gives the same series once we realize that $\frac{1+(-1)^n}{2}$ alternates between the values 1 and 0. But the first way of writing the series is more convenient for using the ratio test, which gives

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\right| = |x^2| \tag{9}$$

and the condition $|x^2| < 1$ gives that the radius of convergence is 1.

3. (4 points) Find the Maclaurin series for $f(x) = a^x, a > 0$ using the definition of a Maclaurin series and determine its radius of convergence.

Solution: The key is to write $f(x) = e^{(\ln a)x}$. This makes it clear that $f'(x) = (\ln a)e^{(\ln a)x}$, $f''(x) = (\ln a)^2 e^{(\ln a)x}$, and in general

$$f^{(n)}(x) = (\ln a)^n e^{(\ln a)x}. \tag{10}$$

After substituting $x = 0$ we get $f^{(n)}(x) = (\ln a)^n$, hence the Maclaurin series of f is

$$f(x) = \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} x^n. \tag{11}$$

To determine the radius of convergence we can use the ratio test, getting

$$\lim_{n\to\infty} \left| \frac{(\ln a)^{n+1} x^{n+1} n!}{(\ln a)^n x^n (n+1)!} \right| = \lim_{n\to\infty} \left| \frac{(\ln a)x}{n+1} \right| = 0 \tag{12}$$

so the radius of convergence is ∞.