Show your work and box your final answer. If you run out of room, continue on the back.

Name and section: ______________________________

1. (4 points) I want to approximate the integral

$$\int_1^5 \frac{1}{x}\,dx \tag{1}$$

using the trapezoid rule with n subdivisions. How large do I need to make n so that the error in this approximation is less than 0.0001? (You do not need to compute the approximation.)

Solution: Recall that the error bound for the trapezoid rule reads

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \tag{2}$$

where n is the number of subdivisions, $[a, b]$ is the interval over which we're integrating, and K is a number such that $|f''(x)| \leq K$ for all $x \in [a, b]$. Here $b - a = 4$, and to find K we should first compute

$$f'(x) = -\frac{1}{x^2}, f''(x) = \frac{2}{x^3} \tag{3}$$

and then figure out how we can bound $|f''(x)|$ on the interval $[1, 5]$. Happily, $f''(x)$ is a decreasing and positive function on this interval, so $f''(1) = 2$ is an upper bound, and we can take $K = 2$. This gives an error bound of

$$|E_T| \leq \frac{2(4)^3}{12n^2} = \frac{32}{3n^2} \tag{4}$$

so we want to find n such that

$$\frac{32}{3n^2} \leq \frac{1}{10000}. \tag{5}$$

Rearranging, we want to solve $320000 \leq 3n^2$, so $n \geq \sqrt{\frac{320000}{3}}$. To get a handle on how big this number is, note that $\frac{320000}{3}$ is about 10^5, so its square root is about $\sqrt{10} \times 10^2$, and $\sqrt{10}$ is about 3, so n is about 300, but a little larger. It's not hard to see that we can take $n = 400$, and with a little more work we can take $n = 330$, since

$$3(330)^2 = 3(108100) = 324300 > 320000 \tag{6}$$

as desired. (Using a calculator we see that the smallest we can take n given the above is $n = 327$.)

2. (3 points) Determine whether the integral $\int_1^3 \frac{1}{\sqrt{x-1}}\,dx$ converges or diverges. (You do not need to evaluate the integral if it converges.)

Solution: The interesting thing is to figure out what the integrand is doing as x approaches 1, since there's a singularity there. To do this we can use the substitution $u = x - 1$ to rewrite the integral as

$$\int_0^2 \frac{1}{\sqrt{u}}\,du \tag{7}$$

which converges because $\frac{1}{2} < 1$. In more detail, the above integral is

$$\begin{aligned}
\lim_{L\to 0^+} \int_L^2 u^{-1/2}\,du &= \lim_{L\to 0^+} \left(2u^{1/2}\right)\Big|_L^2 &(8)\\
&= \lim_{L\to 0^+} \left(2\sqrt{2} - 2\sqrt{L}\right) &(9)\\
&= 2\sqrt{2}. &(10)
\end{aligned}$$

3. (3 points) Determine whether the integral $\displaystyle\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges or diverges. (You do not need to evaluate the integral if it converges.)

Solution: Here the integrand has no singularities, so the interesting thing is to figure out what the integrand is doing as x approaches ∞. When x gets large the 1 becomes negligible and the integrand behaves like $\frac{1}{x^{3/2}}$, so the integral should converge by comparison and the p-test for $p = \frac{3}{2}$. More formally, we can write

$$\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,dx = \int_0^1 \frac{1}{\sqrt{x^3+1}}\,dx + \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx \tag{11}$$

where the first integral converges because it is an ordinary integral and we can use the comparison test on the second integral: $0 \le \frac{1}{\sqrt{x^3+1}} \le \frac{1}{x^{3/2}}$, so

$$0 \le \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx \le \int_1^\infty \frac{1}{x^{3/2}}\,dx < \infty \tag{12}$$

as desired.