[W]hen you do have a deep understanding, you have solved the problem and it is time to do something else. This makes the total time you spend in life reveling in your mastery of something quite brief. One of the main skills of research scientists of any type is knowing how to work comfortably and productively in a state of confusion.

Name and section: ______________________________

1. (3 points) Solve $y' - 2y = 0, y(0) = 1$ using power series.

Solution: We have

$$\begin{aligned} y &= \sum_{n\geq 0} a_n x^n && (1)\\ y' &= \sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 0}(n+1)a_{n+1}x^n && (2)\end{aligned}$$

and hence

$$\begin{aligned} y' - 2y &= \sum_{n\geq 0}(n+1)a_{n+1}x^n - 2\sum_{n\geq 0} a_n x^n && (3)\\ &= \sum_{n\geq 0}\left((n+1)a_{n+1} - 2a_n\right) && (4)\\ &= 0. && (5)\end{aligned}$$

It follows that $(n+1)a_{n+1} = 2a_n$ for all $n \geq 0$, or

$$a_{n+1} = \frac{2a_n}{n+1}. \tag{6}$$

This gives $a_1 = \frac{2a_0}{1}, a_2 = \frac{2^2 a_0}{1\cdot 2}, a_3 = \frac{2^3 a_0}{1\cdot 2\cdot 3}, a_4 = \frac{2^4 a_0}{1\cdot 2\cdot 3\cdot 4}$, and in general

$$a_n = \frac{2^n a_0}{1\cdot 2\cdot \ldots \cdot n} = \frac{2^n a_0}{n!}. \tag{7}$$

We conclude that

$$y = a_0 \sum_{n\geq 0} \frac{2^n x^n}{n!} = a_0 e^{2x} \tag{8}$$

exactly as expected. Substituting $y(0) = 1$ gives $a_0 = 1$, so our final answer is

$$y = \sum_{n\geq 0} \frac{2^n x^n}{n!} = e^{2x}. \tag{9}$$

2. (3 points) Solve $y'' + 4y = 0, y(0) = 1, y'(0) = 0$ using power series.

Solution: We have

$$y = \sum_{n\geq 0} a_n x^n \tag{10}$$

$$y' = \sum_{n\geq 1} n a_n x^{n-1} \tag{11}$$

$$y'' = \sum_{n\geq 2} n(n-1)a_n x^{n-2} = \sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n \tag{12}$$

$$\tag{13}$$

and hence

$$y'' + 4y = \sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n + 4\sum_{n\geq 0} a_n x^n \tag{14}$$

$$= \sum_{n\geq 0}\left((n+2)(n+1)a_{n+2} + 4a_n\right)x^n \tag{15}$$

$$= 0. \tag{16}$$

It follows that $(n+2)(n+1)a_{n+2} + 4a_n = 0$ for all $n \geq 0$, or

$$a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}. \tag{17}$$

For the even terms we get $a_2 = \frac{-4a_0}{2\cdot 1}, a_4 = \frac{(-4)^2 a_0}{4\cdot 3\cdot 2\cdot 1}$, and in general

$$a_{2k} = \frac{(-4)^k a_0}{(2k)!}. \tag{18}$$

For the odd terms we get $a_3 = \frac{-4a_1}{3\cdot 2}, a_5 = \frac{(-4)_1^a}{5\cdot 4\cdot 3\cdot 2}$, and in general

$$a_{2k+1} = \frac{(-4)^k a_1}{(2k+1)!}. \tag{19}$$

Altogether this gives

$$y = a_0 \sum_{k\geq 0} \frac{(-4)^k x^{2k}}{(2k)!} + a_1 \sum_{k\geq 0} \frac{(-4)^k x^{2k+1}}{(2k+1)!} = a_0 \cos 2x + a_1 \sin 2x \tag{20}$$

exactly as expected. Substituting $y(0) = 1$ gives $a_0 = 1$ and substituting $y'(0) = 0$ gives $a_1 = 0$, so our final answer is

$$y = \sum_{k\geq 0} \frac{(-4)^k x^{2k}}{(2k)!} = \cos 2x. \tag{21}$$

3. (4 points) Solve $xy'' - y = 0, y(0) = 0, y'(0) = 1$ using power series.

Solution: This time we won't be able to recognize the answer. We have

$$y = \sum_{n\geq 0} a_n x^n \tag{22}$$

$$y' = \sum_{n\geq 1} n a_n x^{n-1} \tag{23}$$

$$y'' = \sum_{n\geq 2} n(n-1) a_n x^{n-2} \tag{24}$$

$$xy'' = \sum_{n\geq 2} n(n-1) a_n x^{n-1} = \sum_{n\geq 1} (n+1) n a_{n+1} x^n \tag{25}$$

and hence

$$xy'' - y = \sum_{n\geq 1} (n+1) n a_{n+1} x^n - \sum_{n\geq 0} a_n x^n \tag{26}$$

$$= \sum_{n\geq 1} \left((n+1) n a_{n+1} - a_n\right) x^n - a_0 \tag{27}$$

$$= 0. \tag{28}$$

It follows that $a_0 = 0$ and $(n+1)na_{n+1} = a_n$ for $n \geq 1$, or

$$a_{n+1} = \frac{a_n}{n(n+1)}. \tag{29}$$

This gives $a_2 = \frac{a_1}{1\cdot 2}, a_3 = \frac{a_1}{(1\cdot 2)\cdot(2\cdot 3)}, a_4 = \frac{a_1}{(1\cdot 2)\cdot(2\cdot 3)\cdot(3\cdot 4)}$, and in general

$$a_n = \frac{a_1}{(1\cdot 2)\cdot(2\cdot 3)\cdot \ldots \cdot((n-1)\cdot n)} = \frac{n a_1}{(n!)^2}. \tag{30}$$

Altogether this gives

$$y = a_1 \sum_{n\geq 1} \frac{n x^n}{(n!)^2}. \tag{31}$$

Substituting $y'(0) = 1$ gives $a_1 = 1$, so our final answer is

$$y = \sum_{n\geq 1} \frac{n x^n}{(n!)^2}. \tag{32}$$